Soggy Jogging in Flatland:
A 2D Analysis of Running in the Rain

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Have you ever been caught out in the rain and wondered whether to run or walk? In an all out downpour, there doesn't seem to be much question. You will be totally soaked in very short order, so the natural inclination is to reach shelter as fast as you can. But in a lighter rain it is not so obvious. If you go faster, maybe you sweep up more rain with your front. Is it possible to stay drier by going slower?

We want to share a beautiful geometric analysis of this situation that dispels the clouds of doubt and shines bright sunshine on the question. The geometry is so compelling that you will agree immediately: in most conditions the best strategy is to run at full speed. But we do need a few assumptions, naturally, and to simplify the story we shall take a field trip to Flatland and investigate two dimensional running in the rain.

So consider Mr. Pictogram, pictured in Figure 1, traveling through Flatland in a light rain. Let's assume that the rain is falling steadily at a constant rate and in a constant vertical direction. If you could freeze time at one instant, the air would be full of individual raindrops. We will assume that they are of equal size and uniformly distributed. That means that the total amount of water in any region is proportional to the area of the region.

In some cases, the logic of matching your pace to the wind just doesn't hold water.

Say we freeze time at the instant Mr. Pictogram begins his trip. Visualize him starting off, surrounded by raindrops, suspended motionless. Let's color all the raindrops that will eventually fall on him as he moves forward. These colored drops occupy a region of the plane that we shall call the *rain region*. By our earlier observation, the total amount of water in the rain region is proportional to its area. And that is the amount of water that will fall on Mr. P.

As we shall see, the size and shape of the rain region depend on how fast Mr. P moves. At a casual stroll, Mr. P might have a rain region of area S. Traveling at a brisk jog Mr. P will have a different rain region J. If the area of J is twice as large as the area of S, then Mr. P will collide with twice as much water when jogging as when strolling. In general, we can compare different ways to move through the rain by comparing the areas of the corresponding rain regions.

Shapes of Rain Regions

To get a feel for the nature of rain regions, it helps to consider a couple of extreme cases. First, suppose Mr. P travels at super speed—so fast that he reaches shelter before the rain drops have a chance to fall at all. In this case, the rain region stretches out in front of him, as shown in Figure 2. Mr. P sweeps out all the rain drops suspended in his path for how ever far he runs. In this case we find the rain region by sweeping Mr. P horizontally.

At the opposite extreme, Mr. P doesn't move at all. He runs so slowly that he is effectively standing still. In this case,

Figure 1. Mr. Pictogram jogs in the rain.

Figure 2. Rain region when Mr. P runs at super speed.

the rain region extends directly above him, as shown in Figure 3. The height of the region depends on how long Mr. P remains standing there. For a fixed amount of time, the rain region will extend a fixed distance vertically, as illustrated. Here we find the rain region by sweeping Mr. P vertically. These two extreme cases help us see what the rain region looks like traveling at an intermediate speed. The rain

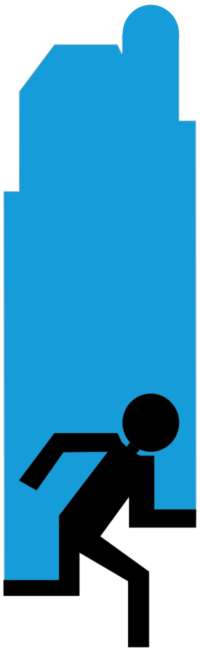

Figure 3. Rain region when Mr. P stands still.

drops immediately above Mr. P fall on him at the start of his trip. But raindrops slightly higher initially will be the ones that hit him a little later in the trip. As he moves forward and time passes, the raindrops he encounters have fallen for progressively longer periods from their initial positions. In this case the rain region is found by sweeping Mr. P at an angle, as shown in Figure 4. The slope of the angle depends on Mr. P's speed and the speed at which the rain is falling. If Mr. P goes super fast, then the slope is 0. If he goes infinitely slowly, the slope is infinite. If he goes at exactly the speed of the rain's descent, the slope is 1. In general, we sweep out the rain region at an angle whose tangent is the ratio of the rain's rate of descent to Mr. P's rate of travel.

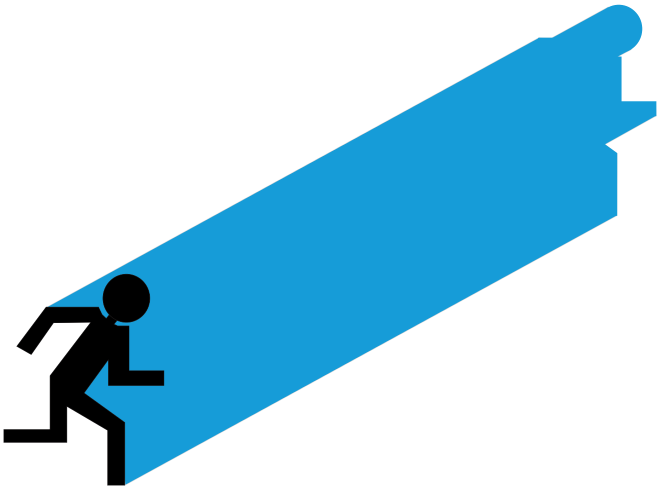

Figure 4. Rain region when Mr. P travels at a normal rate.

Minimizing Exposure to the Rain

Now let's use the idea of rain regions to analyze Mr. P's options. In Figure 5 Mr. P is shown running a distance d. To simplify the geometry he is idealized to a rectangular shape of height h and width w. The rain region, swept out at an angle α that depends on P's speed, can thus be decomposed into two parallelograms. The more lightly shaded parallelogram holds all the rain drops that will collide with Mr. P's front. The area of that part is hd and is independent of the angle α. The darker parallelogram holds all the water that will fall on Mr. P's top. A routine application of trigonometry shows that this part of the rain region has area $wd \tan \alpha$. To minimize the area of the entire rain region, Mr. P should go as fast as possible because that will minimize $\tan \alpha$. Notice that the limiting cases are consistent with these conclusions. At infinite speed, α is zero. Mr. P will sweep out all the rain in front of him, but not a drop will fall on him from above. In this case the area of the darker parallelogram is 0, as the formula $wd \tan \alpha$ predicts. At the other extreme, when Mr. P stands still, α is $\pi/2$, so $\tan \alpha$ goes to infinity. This time both the geometry and the formula indicate an infinite rain region.

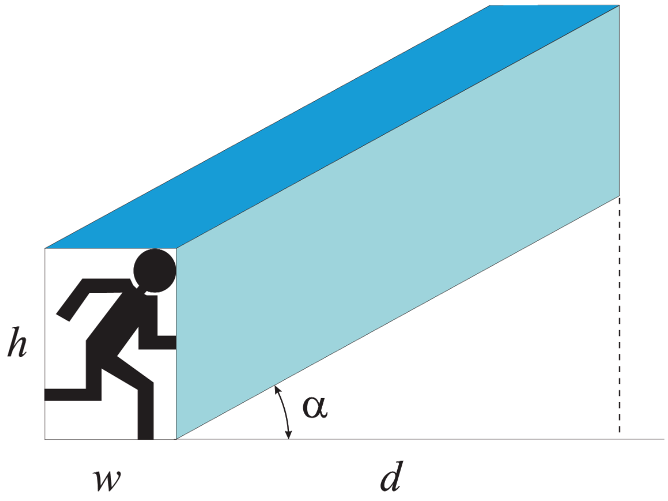

Figure 5. Decomposed rain region.

The Vexing Tail Wind and Mr. Circle

Things get more interesting if we adjust one of our assumptions and allow the wind to blow. To keep it simple, let's assume that there are no gusts. The steady wind causes the rain to fall at a fixed angle. What happens if, as before, Mr. P moves in the positive x direction, but now there is a tail wind? Historically this situation has proved vexing, and the mathematical and meteorological literature is sprinkled with assertions, retractions, and corrections. In the October, 2009 issue of *Mathematics Magazine* we have an article titled "Keeping Dry: The Mathematics of Running in the Rain," which explores the mathematical history of this topic in some depth. The conventional wisdom is that Mr. P is best served by traveling at the precise speed of the horizontal component of the tail wind, for in this situation the *apparent* rain direction is vertical and so the rain region lies directly overhead. For a rectangular traveler, and for a sufficiently strong tail wind, the speed of the tail wind is indeed the optimal pace. Why, you may wonder, must the tail wind be sufficiently strong? Well, consider an extreme case: in a tail wind of, say, one inch per hour, Mr. P would *have* to get wetter crawling like a tortoise than running like a hare simply because he would be exposed to the elements hundreds of times longer. In fact, in the ultimate extreme case, with a tail wind of 0, the logic of matching your pace to the wind just doesn't hold water.

So Mr. P should only keep pace with the wind if it is going sufficiently fast. And how fast might that be? The answer can be worked out exactly, as shown in several of the references to our *Mathematics Magazine* article.

More surprising to us was the discovery that the shape of the traveler is paramount in these calculations. To illustrate this idea, consider Mr. Circle, a solid citizen in Flatland, always ready to do a good turn. He is special because his rain region always has the same shape: a rectangle with half a circle cut out of one end and pasted onto the other (see Figure 6).

This means the area of the rain region, regardless of his speed or the speed of the tail wind, is always equal to the area of a rectangle of width $2r$, where r is the radius of Mr. C. The length of the rectangle depends on his speed, the wind speed, and how far he travels. For the purpose of illustration, suppose he moves one unit in the positive x direction at a speed of s units per second. The duration of his trip is then $1/s$ seconds. Now suppose that there is a 5 unit per second tail wind and that the rain is falling with a vertical velocity component of 3 units per second. Then in $1/s$ seconds, a drop of rain will travel along the vector $\langle 5/s, -3/s\rangle$. At the same time, Mr. C travels along the vector $\langle 1,0\rangle$. So, from Mr. C's perspective, the rain appears to be traveling along the *apparent rain vector* $\langle 5/s - 1, -3/s\rangle$.

Figure 6. Mr. Circle's rain region can be thought of as a rectangle.

Since the rain region is exactly as long as this vector and is $2r$ units wide, it has area

$$\frac{2r\sqrt{(5-s)^2+3^2}}{s}$$

We would like to minimize this as a function of s. Differentiation with respect to s reveals exactly one critical point, at which the area of the rain region is minimal: $s = (5^2 + 3^2)/5 = 6.8$. Note that this is faster than the 5 unit per second tail wind!

Generalizing to a tail wind of w units per second and rain falling with a vertical velocity component of v units per second, a similar analysis shows that Mr. C's optimal speed is $s = (w^2 + v^2)/w$, and since $v > 0$, that always *exceeds* w, the speed of the tail wind. This means that for Mr. Circle, traveling a bit faster than the speed of the tail wind will *always* keep him driest. Rectangular travelers moving with a tail wind, by contrast, are best served by either running full speed (if the wind is weak) or by traveling at precisely the speed of the tail wind (if it is sufficiently strong). There is no getting around the fact that shape matters!

Back to the Real World

The adventures of Messers P and C are all well and good, but what lesson do they hold for those of us who don't live in Flatland? The ideas that govern running in Flatland all extend in a natural way to three dimensions. Rain regions, wind directions, tail winds, and runner shapes all play a role, with the additional consideration of the effect of a *cross wind*. The complete story can be found in our *Mathematics Magazine* article. As is the case in Flatland, running hard is often best. But under some very special circumstances the optimal strategy to stay as dry as possible is *not* to run at full speed. Given the right combination of tail and cross winds, you can actually do a little better at something less than your fastest possible pace.

An interactive demonstration, in which wind conditions and the shape of a three-dimensional traveler can be manipulated in real time, is available for download at www.maa.org/mathhorizons/supplemental.htm.

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