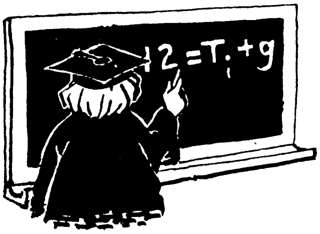

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

Sum Formulas for Sine and Cosine

Dan Kalman, University of Wisconsin, Green Bay, WI

One of the treasures of antiquity is a diagram which quickly yields a proof of the Pythagorean Theorem. In a similar spirit, Figure 1 can be used to prove some other important results in trigonometry. It remains only to indicate proper labeling. Since the diagonal of the inscribed rectangle is 1, its sides measure $\cos\beta$ and $\sin\beta$. These constitute hypotenuses for the similar triangles in the four corners of the large rectangle. Therefore, looking at triangle ABC in Figure 2, we have:

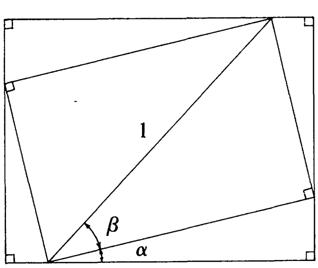

Figure 1.

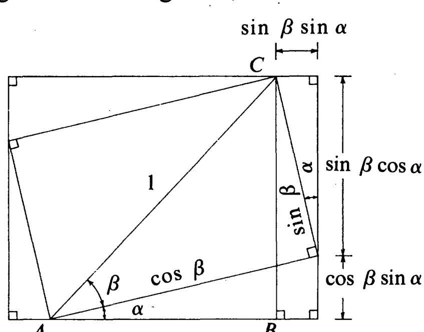

Figure 2.

$$\sin(\alpha+\beta) = \cos\beta\sin\alpha + \sin\beta\cos\alpha$$

and

$$\cos(\alpha+\beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha.$$

Figure 2 is valid for acute α and β with sum less than 90°, the only domain that is meaningful for the trigonometry of right triangles. However, adopting the usual interpretation of reference angles, we can modify Figure 2 to establish the addition formulas in general for acute β. Clearly, the inscribed rectangle may be constructed for any acute β. Hold the lower left-hand corner of this rectangle at the center A of the unit circle and rotate through any angle α so that the bottom of the rectangle becomes the terminal side. Then construct the circumscribing rectangle with sides parallel to the coordinate axes. Figure 3 illustrates one such situation with angle $\alpha + \beta$ in the third quadrant. As before, the dimensions of the small triangles in the corners are simply related to α', the reference angle for α. Thus, these dimensions may be specified in terms of α as in Figure 3. Now the coordinates of C are determined, giving the sine and cosine of $\alpha + \beta$. For other orientations of the inscribed rectangle, similar arguments may be used.

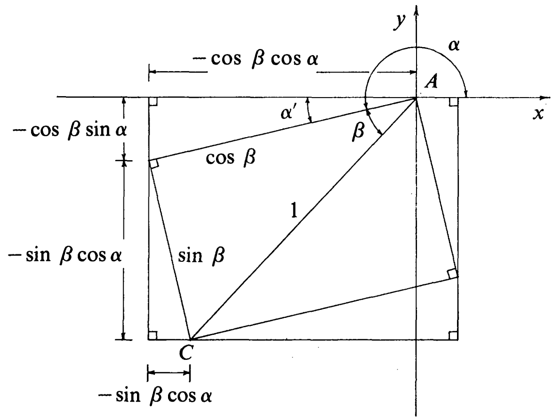

Figure 3.

As a special case, take β to be the complement of α. Then Figure 2 becomes Figure 4. Thus, we obtain the identity $\sin^2\alpha + \cos^2\alpha = 1$, and a proof of the Pythagorean theorem follows.

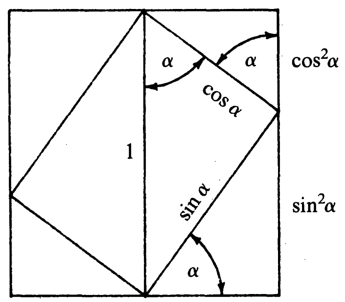

Figure 4.