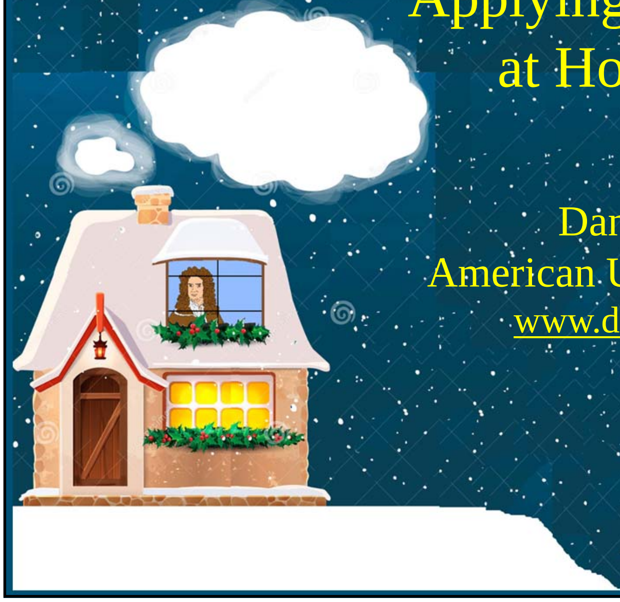
Newton Cooling in the Attic: Applying ODEs at Home



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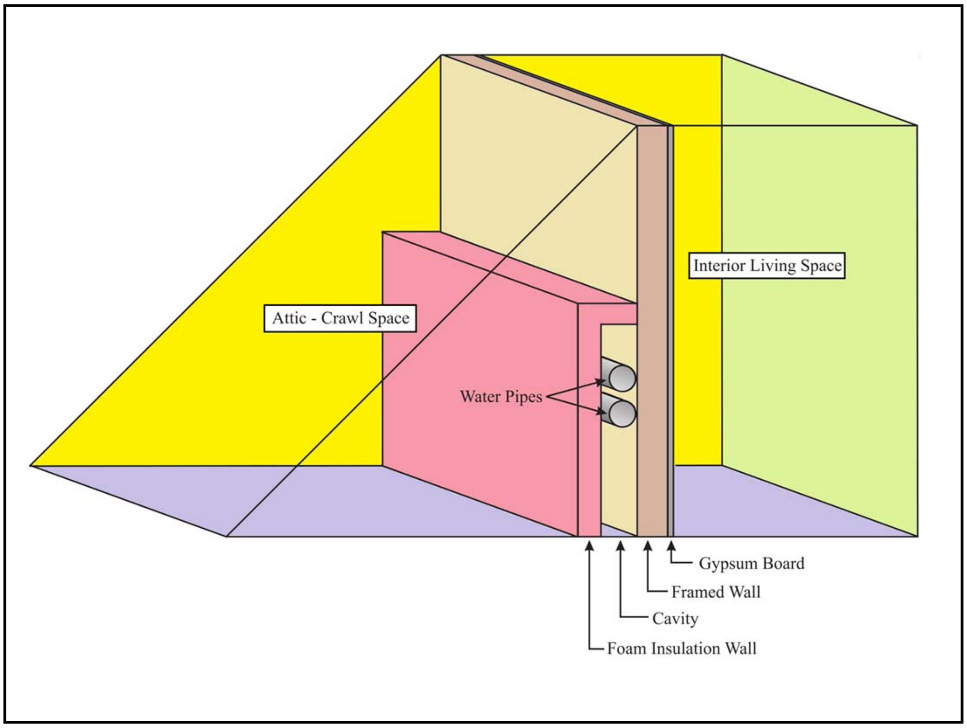
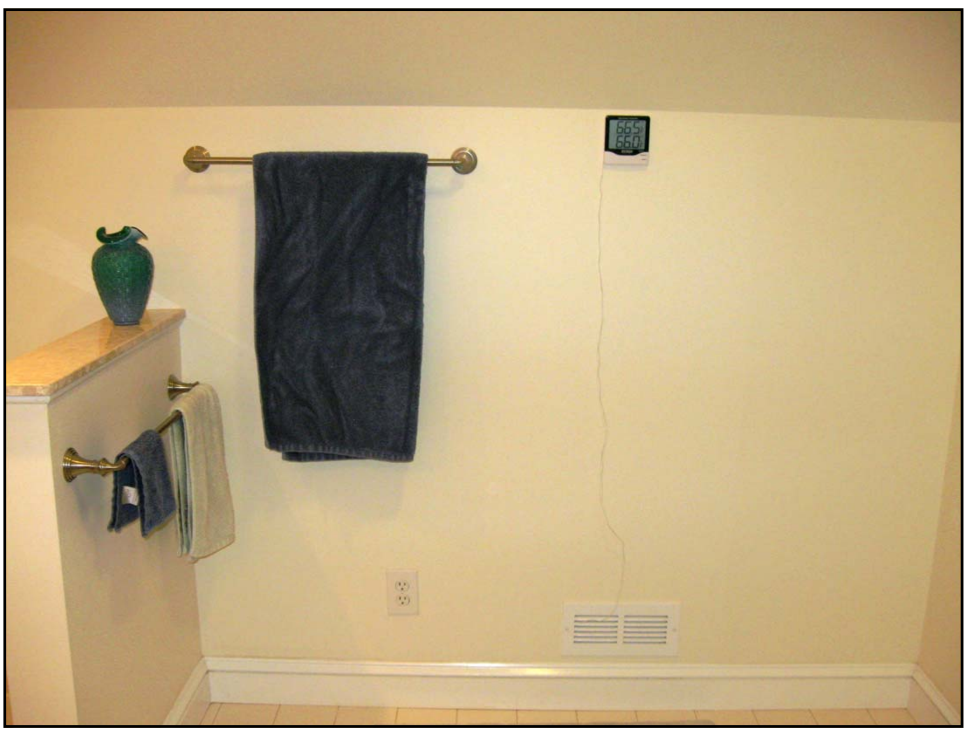
Cold Air Problem

- Water pipes run through an attic crawl space
- Cold air infiltration under high wind conditions
- Solution: create insulated enclosure around the pipes
- Question: How cold a sustained temperature can be tolerated in the crawl space w/o risking frozen pipes?


Interior Living Space
Attic - Crawl Space
Water Pipes
Gypsum Board
Framed Wall
Cavity
Foam Insulation Wall



ODE Model

- Assume Newton cooling with two ambient temperatures
- The cavity interfaces with two thermal reservoirs, each at a constant temperature
- Assume the rate of heat flow between each reservoir and the cavity is proportional to the temperature difference
- Different proportionality constants for the two reservoirs
- The constant for heat flow into the cavity is greater than the constant for heat flow out of the cavity

Example

Assumptions

- Ambient temperature in the heated room is 70°
- Ambient temperature in the crawl space is 10°
- T represents the temperature in the cavity
- Proportionality constant for heat flow into cavity is r
- Constant for heat flow out of cavity is s

Differential Equation:

$$\frac{dT}{dt} = r(70 - T) - s(T - 10)$$

Solution

$$\frac{dT}{dt} = r(70 - T) - s(T - 10)$$
$$= -(r + s)T + 70r + 10s$$
$$= -(r + s)\left(T - \frac{70r+10s}{r+s}\right)$$

This is a standard Newton cooling problem with rate constant $r + s$ and ambient temperature $\frac{70r+10s}{r+s}$.

Solution: $T = Ae^{-(r+s)t} + \frac{70r+10s}{r+s}$; $A = T_0 - \frac{70r+10s}{r+s}$

Equilibrium: $T_\infty = \frac{70r+10s}{r+s}$

Interpretation

- Assume initial cavity temperature is between the two ambient temperatures $(10 < T_0 < 70)$
- Temperature in cavity will asymptotically approach $T_\infty = \frac{70r+10s}{r+s}$
- Note this is a weighted average of the two ambient temperatures: $T_\infty = 10\frac{s}{r+s} + 70\frac{r}{r+s}$
- This is between 10 and 70, with weights in the ratio $s : r$.
- Example: $r = 5s$. Divide the interval $[10,70]$ into sixths, $T_\infty = 10\frac{1}{6} + 70\frac{5}{6}$ so is 1/6 of the way from 70 to 10. *ie.* $T_\infty = 60$

In General

- Assume initial cavity temperature is between the two ambient temperatures $(C < T_0 < H)$
- Temperature in cavity will asymptotically approach $T_\infty = \frac{Hr+Cs}{r+s}$
- This equals $\frac{Hr/s+C}{r/s+1}$, so T_∞ depends only on the ratio $\frac{r}{s}$
- This also leads to $\frac{r}{s} = \frac{T_\infty - C}{H - T_\infty}$ showing again how the equilibrium value divides the interval from *C* to *H* into parts with ratio $r : s$

Rate constants and *R* values

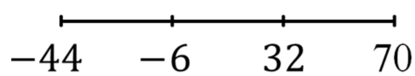
- Accurate modeling of temperature response to heat flow is complicated. For our purposes we use zeroth order approximations: constant rate constants, heat capacities, etc.
- Change in temperature, ΔT, is proportional to increase in energy ΔE (cals, btus, etc).
- $\Delta E \propto \frac{T-T_0}{R} A\Delta t$ where *R* is the insulation's *R* value, *A* is area of the boundary through which heat flows
- Newton cooling rate constant $\propto \mathrm{A}\frac{1}{R}$
- $\frac{r}{s} = \frac{\mathrm{A}_r}{\mathrm{A}_s}\frac{R_s}{R_r} \cong \frac{R_s}{R_r}$ if $\mathrm{A}_r \cong \mathrm{A}_s$

Rate constants and *R* values

- $\frac{r}{s} \cong \frac{R_s}{R_r}$ if $A_r \cong A_s$
- *R* value for 2 inch foam insulation is about 10, for 1/2 inch dry wall it is about 0.5.
- $\frac{r}{s} \cong \frac{10}{.5} = 20$
- With transfer grill in warm wall, R_r value will be less than 0.5, increasing $\frac{r}{s}$ estimate by unknown amount.
- $A_r < A_s$, decreasing $\frac{r}{s}$ estimate by less than 50%
- This suggests $\frac{r}{s} \geq 10$

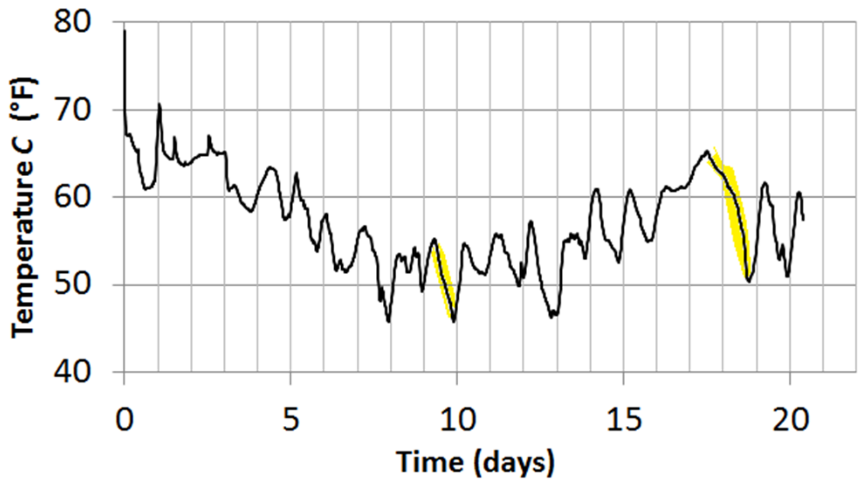
Original Question

- How cold can *C* be without risking frozen pipes?
- Assume H = 70
- Assume (conservatively) $\frac{r}{s} = 5$
- Want $T_\infty = \frac{5H+C}{5+1} = \frac{350+C}{6} > 32$
- $C > 32 \cdot 6 - 350 = -158$ (!)
- Even more conservative: if $\frac{r}{s} = 2$ the minimal tolerable value is $C = -44$
- Note 32 is 1/3 of the way from 70 to -44



Model Validation

- Ambient temp in crawl space, C, is not constant



Model Validation

- Ambient temp in crawl space, C, is not constant
- ODE still solvable for $C(t)$ linear, exponential, etc
- $C(t)$ Linear: $T(t) = \alpha e^{-(r+s)t} + W(t) + \beta$ where $W(t) = \frac{Hr+C(t)s}{r+s} = \frac{Hr/s+C(t)}{1+r/s}$ (weighted average of H & $C(t)$)
- Try to fit this model to observed data when $C(t)$ is roughly linear. Parameters $\alpha, \beta, r+s, r/s$
- If fit is good, can find effective r and s
- Need simultaneous measurements of temps in warm room, cavity, and crawl space
- Still struggling with instrumentation

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- Thanks
- Questions?