Polynomia Pasttimes

Activities and Diversions from the Province of Polynomia

Dan Kalman
American University

www.dankalman.net/mathfest11

Reminder: What's a polynomial?

$$5x^3 - 7x^2 + 3x - 2$$

$$15x^2 - 14x + 3$$

$$5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Polynomials have ROOTS

$$5x^3 - 7x^2 + 3x - 1 = 0$$

$$x = 1$$

Roots are related to *Factors*

$$5x^3 - 7x^2 + 3x - 1 = (x-1)(5x^2 - 2x + 1)$$

Complete factorization …

$$5x^3 - 7x^2 + 3x - 1 = (x-1)\left(x - \frac{2+\sqrt{-16}}{10}\right)\left(x - \frac{2-\sqrt{-16}}{10}\right)$$

$$= (x-1)(x-.2-.4i)(x-.2+.4i)$$

Finding Roots is Hard …

… But we *can* find out some things easily

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

- The sum of the roots is 11/5
- The average of the roots is 11/20
- The sum of the reciprocals of the roots is 7/3
- We'll get back to roots in a bit
  First let's look at computation
- Can you compute p(3) in your head?

Horner's Form

- Standard descending form

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

- Horner form

$$p(x) = (((5x - 11)x + 6)x + 7)x - 3$$

- Also referred to as partially factored or nested form

Derivation of Horner Form

$$\begin{aligned} p(x) &= 5x^4 - 11x^3 + 6x^2 + 7x - 3 \\ &= (5x^3 - 11x^2 + 6x + 7)x - 3 \\ &= ((5x^2 - 11x + 6)x + 7)x - 3 \\ &= (((5x - 11)x + 6)x + 7)x - 3 \end{aligned}$$

Quick Evaluation

- Compute $p(2)$:
  $(((5 \cdot 2 - 11)2 + 6)2 + 7)2 - 3$
- Answer = 27
- Compute $p(3)$:
  $(((5 \cdot 3 - 11)3 + 6)3 + 7)3 - 3$
- Answer = 180
- Compute $p(2/5)$:
  $(((5 \cdot \# - 11)\# + 6)\# + 7)\# - 3$
- Answer = 23/125?

Getting Back to our Roots

Coefficients and combinations of roots

Product of Roots

Constant term / highest degree coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Product of the roots is …

– 3/5

Key Idea of Proof

- For our example

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

- Say the roots are r, s, t, and u.
- $p(x) = 5(x - r)(x - s)(x - t)(x - u)$
- Multiply this out to find the constant term

$$5rstu = -3$$

Sum of Roots

– 2^{nd} highest degree coefficient divided by highest degree coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Sum of the roots is …

11/5

Average of the roots is …

11/20

Exercises

1. Prove the sum of the roots result

2. Prove this: For any polynomial p , the average of the roots of p is equal to the average of the roots of the derivative p'.

Reverse Polynomial

- Consider the polynomial

$$p(x) = x^6 - 7x^5 + 11x^4 + 13x^3 - 5x^2 + 2x + 1$$

- The reverse polynomial is

$$\text{Rev } p(x) = 1 - 7x + 11x^2 + 13x^3 - 5x^4 + 2x^5 + x^6$$

- Question: How are the roots of Rev *p* related to the roots of *p*?

- Answer: Roots of reverse polynomial are reciprocals of roots of the original.

Sum of Reciprocal Roots

– 1st degree coefficient divided by the constant coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Sum of the reciprocal roots is …

7/3

Average of the reciprocal roots is …

7/12

Exercises

1. Prove this: For any polynomial p of degree n with nonzero constant term,
   Rev $p(x) = x^n\, p(1/x)$.

2. Use 1 to prove that the roots of Rev $p(x)$ are the reciprocals of the roots of $p(x)$.

3. Use 2 to prove: if
   $p(x) = a_n x^n + \cdots + a_1 x + a_0$
   and $a_0 \neq 0$, then the sum of the reciprocals of the roots of p is $-a_1 / a_0$

Polynomial Long Division

- Example: $(x^2 - 5x + 6) \div (x-3)$
- Redo the example working from the constants upward
- Another example: $(x^2 - 5x + 6) \div (x-1)$
- Answer $-6 - x - 2x^2 - 2x^3 - 2x^4 - 2x^5 - \cdots$
- Similar to the long division $1 \div 3$ to find .333333…
- Alternate form of answer:

  $-6 - x - 2x^2 (1 + x + x^2 + x^3 \cdots) = -6 - x - 2x^2/(1-x)$
- Similar to mixed fraction form of answer to a division problem.

Amazing Application

- Start with $p(x) = x^3 - 2x^2 - x + 2$
- Find the derivative (Do it on overhead)
- Reverse both (Do it on overhead)
- Do a long division problem of the reversed p(x) into the reversed p'(x), working from the constants forward

  (Do it on overhead)
- The coefficients have an astonishing interpretation: sums of powers of roots

Checking the answer

- $p(x) = x^3 – 2x^2 – x + 2 = (x-2)(x^2 – 1)$
- Roots are 2, 1, and -1
- Sum of roots = 2
- Sum of squares of roots = 6
- Sum of cubes = 8
- Sum of fourth powers = 18
- Etc.

Proof Hints

- Rev $p(x) = x^n\, p(1/x)$
- Logarithmic Derivative: $f' / f = (\ln f)'$
- If $f(x) = (x - r)(x - s)(x - t) \cdots$ then

$$(\ln f(x))' = \frac{1}{x-r} + \frac{1}{x-s} + \frac{1}{x-t} + \cdots$$

- Geometric Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{and} \quad \frac{1}{x-a} = \frac{-1/a}{1-x/a}$$

Palindromials

- $p(x)$ = reverse $p(x)$
- Example: $x^4 + 7x^3 - 2x^2 + 7x + 1$
- 1 and -1 are not roots, so roots come in reciprocal pairs
- Must factor as $(x-r)(x-1/r)(x-s)(x-1/s)$
- Rewrite: $(x^2 – ux + 1)(x^2 – vx + 1)$ where $u = r+1/r$ and $v = s + 1/s$

Matching Coefficients

- $(x^2 - ux + 1)\,(x^2 - vx + 1) = x^4 + 7x^3 - 2x^2 + 7x + 1$
- $u + v = -7$ and $uv + 2 = -2$
- Two unknowns. Sum = -7, product = -4
- They are the roots of $x^2 + 7x - 4 = 0$
- u and v are given by $\dfrac{-7 \pm \sqrt{65}}{2}$
- Our factorization is

$$\left(x^2 - \frac{-7+\sqrt{65}}{2}x + 1\right)\left(x^2 - \frac{-7-\sqrt{65}}{2}x + 1\right)$$

Solve for x

$$\left(x^2 - \frac{-7+\sqrt{65}}{2}x + 1\right)\left(x^2 - \frac{-7-\sqrt{65}}{2}x + 1\right) = 0$$

- Use quadratic formula on each factor
- Roots from first factor are

$$\frac{1}{2}\left(\frac{-7+\sqrt{65}}{2} \pm \sqrt{\frac{98-14\sqrt{65}}{4}}\right) = \frac{1}{4}\left(-7+\sqrt{65} \pm \sqrt{98-14\sqrt{65}}\right)$$

- Remaining roots are

$$\frac{1}{4}\left(-7-\sqrt{65} \pm \sqrt{98-14\sqrt{65}}\right)$$

General Reduction Method

- $p(x) = ax^6 + bx^5 + cx^4 + dx^3 + cx^2 + bx + a$
- $p(x)= x^3(ax^3 + bx^2 + cx + d + cx^{-1} + bx^{-2} + ax^{-3})$
- $p(x)/x^3 = a(x^3+1/x^3) + b(x^2+1/x^2) + c(x+1/x)+d$
- We want roots of $a(x^3+1/x^3) + b(x^2+1/x^2) + c(x+1/x)+d$
- Almost a polynomial in $u = (x+1/x)$.
- $u^2 = x^2 + 2 + 1/x^2 \rightarrow x^2+1/x^2 = u^2 – 2$
- $u^3 = x^3 + 3x + 3/x + 1/x^3 = x^3 + 3u + 1/x^3$
  $\rightarrow x^3+1/x^3 = u^3 – 3u$
- Leads to a cubic polynomial in u:
  $$a(u^3 – 3u) + b(u^2 – 2) + c(u)+ d$$

Example

$$x^8 + 3x^7 - 6x^6 + 12x^5 - 13x^4 + 12x^3 - 6x^2 + 3x + 1 = 0$$

- Make the standard reduction

$$u^4 + 3u^3 - 10u^2 + 3u + 1 = 0$$

- It's another palindromial! Reduce again

$$v^2 + 3v - 12 = 0$$

- Solve with quadratic formula $v = \dfrac{-3 \pm \sqrt{57}}{2}$
- Find u: $v = u + 1/u$ so $u^2 - vu + 1 = 0$
- Solve for u

$$u = \frac{-3 - \sqrt{57} \pm \sqrt{50 + 6\sqrt{57}}}{4}$$

Solve for x

- We have found 4 values for u
- We know x + 1/x = u
- Solve $x^2 - ux + 1 = 0$ with quadratic formula for each known u value
- That gives 8 roots
- Here is one:

$$\frac{-3-\sqrt{57}+\sqrt{50+6\sqrt{57}}+i\sqrt{(6+2\sqrt{57})\sqrt{50+6\sqrt{57}}-52-12\sqrt{57}}}{8}$$