The Hitchhiker's Guide to Mathematics
MAA Minicourse August 2015

Homework Set K1: Quickies

1. Prove that the sum of the reciprocals of the roots of the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 + a_0$$

equals $-a1/a0$. [Ref: pages 2&3 of quickies handout.]

2. Let $p(x)$ be an n degree polynomial with roots $r_1, r_2, \cdots, r_n$. Define the root power sum sequence

$$\begin{aligned} s_0 &= r_1^0 + r_2^0 + \cdots + r_n^0 \\ s_1 &= r_1^1 + r_2^1 + \cdots + r_n^1 \\ s_2 &= r_1^2 + r_2^2 + \cdots + r_n^2 \\ &\vdots \end{aligned}$$

Prove that

$$\frac{\text{reversed } p'(x)}{\text{reversed } p(x)} = s_0 + s_1 x + s_2 x^2 + \cdots.$$

Hints and Comments.

- The right side of the desired identity can be rewritten as the sum of n series, one for each root r_k. The kth such series is $1 + r_k x + r_k^2 x^2 + r_k^3 x^3 + \cdots$.
- Recall that for any polynomial $q(x)$ of degree m the reverse polynomial is $x^m q(1/x)$.
- In general, for any differentiable function $q(x)$, the ratio $q'(x)/q(x)$ is the derivative of the natural logarithm of q. If $q(x)$ is a product of factors $(x - t_k)$, then the logarithm is the sum of terms $\ln(x - t_k)$.

3. Let $p(x)$ be an n degree polynomial with roots $r_1, r_2, \cdots, r_n$. As before, define the root power sum sequence

$$\begin{aligned} s_0 &= r_1^0 + r_2^0 + \cdots + r_n^0 \\ s_1 &= r_1^1 + r_2^1 + \cdots + r_n^1 \\ s_2 &= r_1^2 + r_2^2 + \cdots + r_n^2 \\ &\vdots \end{aligned}$$

A root of p is said to be *dominant* if it has maximal absolute value among the roots of p. For the questions below, assume that p has a unique dominant root r.

A. Explain why, for sufficiently large m, s_m is approximately equal to r^m.

B. Show that $\sqrt[m]{s_m} \to r$ as $m \to \infty$.

C. Consider the polynomial $p(x) = x^5+3x^4-6x^3-2x+8$. Using the long division algorithm, find the first 5 terms of the sequence s_m. Then compute the corresponding values of $\sqrt[m]{s_m}$. Then use these values to examine the behavior of the sequence $p(\sqrt[m]{s_m})$.

Newton mentions this as a possible application of his identities, though he gives an example only for a cubic.[1] A closely related method is to consider the ratios s_{m+1}/s_m, which again converge to the dominant root. This is a special case of a method developed by Daniel Bernoulli.[2] These historical references are from an unpublished paper by Stacy Langton.

[1]Isaac Newton, *Universal Arithmetick: or, a Treatise of Arithmetical Composition and Resolution*, 1728, in *The Mathematical Works of Isaac Newton*, edited by D. T. Whiteside, Johnson Reprint Corporation, 1967, vol. 2, 3–134.

[2]Daniel Bernoulli, "Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usu pro inveniendis radicum omnium aequationum algebraicarum ostenditur", *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, vol. 3, 1728 (1732), 85–100. Reprinted in *Die Werke von Daniel Bernoulli*, vol. II, Birkhäuser, 1982, 49–64.