The Hitchhiker's Guide to Mathematics
MAA Minicourse August 2015

Postcard Comments

Postcard Number 1. Let $x_k = \cos(10^k)$ for $k = 1, 2, \ldots$. Show that $\{x_k\}$ is a convergent sequence — when the argument is interpreted to be in degrees.

Comment. In fact, the sequence is eventually constant, because 10^k is eventually constant mod 360. For $k \geq 3$ note that 360 is a divisor of $10^{k+1} - 10^k = 10^k \cdot 9 = 10^{k-3} \cdot 9000$. Thus for all such k, $x_k = \cos(1000) = \cos(-80) = \sin(10)$, which is a root of $8x^3 - 6x + 1 = 0$, and hence irrational.

An interesting related question is: for what integral values of b is the sequence b^k eventually constant mod 360? On the other hand, it appears that when $x/(2\pi)$ is irrational, the points on the unit circle with polar angle x^k are almost always uniformly distributed, and so nonconvergent. A reference is the article on Equidistributed Sequences at Mathworld (http://mathworld.wolfram.com/EquidistributedSequence.html) which attributes a result on equidistribution of powers to Hardy and Littlewood.

Postcard Number 2. How big is the angle in each point of a 7 pointed star? Actually, there are two different 7 pointed stars possible. In the case of the star with the larger angle in each point, what is the measure in degrees of the angle?

Comment. Routine geometry shows that the answer is 540/7 degrees. But expressing this as a mixed fraction we find an amazing result: each point of a 7 pointed star (of the second kind) has an angle of $77\frac{1}{7}$ degrees! Is there any other n such that an n pointed star has angles of $nn\frac{1}{n}$ degrees?

Postcard Number 3. With a calculator in degrees mode, compute $\sin(555555^{-1})$ and look at the answer. Repeat for some other number of 5s. Explain.

Comment. The display seems to be a shifted decimal version of π. This is a consequence of three familiar ideas: the conversion of degrees to radians, the small angle approximation for sine, and the decimal expansion of 5/9. Put them all together as follows. First, $555555 = .555555 \cdot 10^6 \approx 10^6(5/9)$. Therefore, $555555^{-1} \approx 10^{-6}(9/5)$. Converting from degrees to radians, we get an angle $\approx 10^{-6}(9/5)(\pi/180) = 10^{-8}\pi$. And that is so small an angle, that it approximates its own sine. That is, measuring angles in degrees, we have $\sin(555555^{-1}) \approx 10^{-8}\pi$. More generally, if we use k 5's, we get an answer that is approximately $10^{-(k+2)}\pi$. Can you determine, as a function of k, how many digits of this approximation will be correct?

Postcard Number 4. How many entries in the nth line of Pascal's Triangle are odd? Here, we count the single 1 at the apex of the triangle as row 0. Thus, for example, row 2 has the entries 1, 2, 1.

Comment. The number of odd entries in row n of Pascal's triangle is equal to $2^{w(n)}$ where $w(n)$ is the number of digits equal to 1 in the binary representation of n. For example, if $n = 13$, the binary representation is 1101, so $w(13) = 3$. The thirteenth row of pascal's triangle is

$$1 \quad 13 \quad 78 \quad 186 \quad 715 \quad 1287 \quad 1716 \quad 1716 \quad 1287 \quad 715 \quad 186 \quad 78 \quad 13 \quad 1$$

and as predicted, there are $2^3 = 8$ odd entries.

See http://mathworld.wolfram.com/PascalsTriangleMod2.html and http://mathworld.wolfram.com/LucasCorrespondenceTheorem.html.

Postcard Number 5. Evaluate the integral

$$\int_{-1}^{1} \left(\sqrt{1-x^2}\right)^n dx$$

where n is a positive integer. Feel free to use Wolfram-Alpha, or the equivalent. Note: there is nothing particularly cute about this problem, but it is relevant to something we'll discuss at the course.

Comment. Not surprisingly, things work out differently for even and odd n. The following results were obtained from Wolfram Alpha:

n	1	2	3	4	5	6
$\int_{-1}^{1} \left(\sqrt{1-x^2}\right)^n dx$	$\frac{\pi}{2}$	$\frac{4}{3}$	$\frac{3\pi}{8}$	$\frac{16}{15}$	$\frac{5\pi}{16}$	$\frac{32}{35}$

There definitely seem to be two patterns, one for odd terms and one for even terms. But they are a bit obscure. It would not necessarily be easy to guess that the formula for the even case, $n = 2k$, is

$$\frac{2^{2k+1}}{(2k+1)\binom{2k}{k}} = 2 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \dots \cdot \frac{2k}{2k+1}.$$