Generalizing a Mysterious Pattern

In his book, *Mathematics: Rhyme and Reason*, Mel Currie discusses what he calls a *mysterious pattern* involving the sequence

$$a_n = 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}$$

where n is the number of nested radicals. The mystery hinges on the fact that a_n converges to a surprising limit. Specifically, $\lim_{n \to \infty} a_n = \pi$. In this paper we explore a variety of extensions of the pattern, sequence, and limit.

It is somewhat surprising how many directions lead off from the mysterious pattern, and how many of them lead to interesting extensions, insights, or generalizations. Here are a few examples:

$$\lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{3}}}} = \frac{2\pi}{3}$$

$$\lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{1 + \varphi}}}} = \frac{4\pi}{5}$$

$$\lim_{n \to \infty} 2^n \sqrt{-2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{16/3}}}} = 2 \ln 3.$$

(Note that ϕ is the golden mean, $(1 + \sqrt{5})/2$.) The basis for this talk is ongoing joint work with Currie.