A Simple Solution of the General Cubic

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The quadratic formula for the general degree two equation is one of the most familiar equations in mathematics. Surely every college mathematics teacher can quote it and derive it without effort. In contrast, the corresponding equation for the solution of the general cubic is quite obscure. We are all aware that such a formula exists, but it is an uncommon mathematician who can quote the result, let alone derive it from first principles. Imagine the surprise, therefore, of discovering a simple algebraic derivation in the middle of looking for something else. Even more surprising, when we reviewed the literature, we discovered (or rediscovered) other derivations that are just as simple. Indeed, Oglesby [9] came up with a closely related approach 75 years ago. In retrospect, the solution of the cubic seems direct enough that we ought to have been more familiar with it. We hope the reader will experience a similar reaction as we share the derivation we found so serendipitously, and sketch the more usual approach.

Before proceeding, we should recall that an arbitrary cubic equation can be reduced to one of the form

$$x^3 + px + q = 0 \tag{1}$$

by a linear change of variable. So in what follows, we will only consider this kind of cubic equation.

The derivation that we will present depends on the following identity.

$$(\omega a + b + c)(a + \omega b + c)(a + b + \omega c) = (a^3 + b^3 + c^3)\omega - 3abc\omega^2 \tag{2}$$

Here, a, b, and c are arbitrary complex constants and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a cube root of 1, and so satisfies a number of identities:

$$\begin{aligned} \omega^3 &= 1 \\ \omega^2 + \omega + 1 &= 0 \\ \omega + 1 &= -\omega^2 \end{aligned}$$

To verify Eq. (2), simply multiply out the left side, collect like monomials in a, b, and c, and apply the identities for ω listed above. Symmetry simplifies the process considerably. Collecting together all terms involving a^2b results in a coefficient of $\omega^2 + \omega + 1 = 0$. By symmetry, the terms involving a^2c, b^2c, etc., also vanish. In the expansion of the left-hand side of the identity, that leaves only terms involving a^3, b^3, c^3, and abc, and by considering the coefficients of these terms, Eq. (2) is easily established.

Identity Eq. (2) was discovered while working on a problem posed in Math Horizons: solve $\sqrt[3]{x+a} + \sqrt[3]{x+b} + \sqrt[3]{x+c} = 0$. Although Eq. (2) turned out to be irrelevant for that problem, the identity was so appealing we were motivated to seek another use for it. In the process, we stumbled on the following simple solution of the general cubic equation.

To render the identity more recognizable, replace a with x, which is to be thought of as the variable of the cubic. That produces

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = (x^3 + b^3 + c^3)\omega - 3xbc\omega^2$$

Factoring out ω on the right and rerranging the remaining terms then leads to

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = \omega(x^3 - 3xbc\omega + b^3 + c^3) \tag{3}$$

Now we can recognize that the right side is essentially the same as what appears in Eq. (1), provided that the following relations hold:

$$-3bc\omega = p \tag{4}$$
$$b^3 + c^3 = q \tag{5}$$

Given values of p and q, we need only determine a b and c satisfying these relations, whereupon Eq. (3) provides a factorization to linear factors. Fortunately, we can solve for b and c in a straightforward way. Indeed, if the original system of equations is rewritten in the following form:

$$\begin{aligned} b^3c^3 &= -p^3/27 \\ b^3 + c^3 &= q \end{aligned}$$

it is immediately apparent that b^3 and c^3 are the roots of the quadratic equation $x^2 - qx - p^3/27 = 0$, and are given by

$$\left[\frac{q \pm \sqrt{q^2 + 4p^3/27}}{2}\right]^{1/3}$$

Note here that when p and q are real, we obtain real values for b^3 and c^3 just when $q^2 + 4p^3/27 \geq 0$. That leads to

$$\begin{aligned} b &= \left[\frac{q + \sqrt{q^2 + 4p^3/27}}{2}\right]^{1/3} \\ c &= \left[\frac{q - \sqrt{q^2 + 4p^3/27}}{2}\right]^{1/3} \end{aligned}$$

When the equations for b^3 and c^3 produce complex (that is, non-real) values, we have to be a bit more careful. There are three complex cuberoots among which to choose b and c, and not all combinations satisfy the original equations for b and c. While it is clear that Eq. (5) will be satisfied in any case, Eq. (4) requires that consistent values of b and c be selected. For this situation, we can choose any of the three complex cuberoots for b, and then define c as $-p/(3\omega b)$.

To complete the solution of the cubic, we note that the solutions to Eq. (1) must also be roots of

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = 0$$

By inspection, the solutions are

$$x = -(b+c)/\omega; \quad x = -(\omega b + c) \quad x = -(b + \omega c)$$

This result is closely related to, but slightly different from the standard solution to the cubic that has been handed down with little if any modification since it was published by Cardano in 1545. Although it was originally derived by a different method, Cardano's solution can be formulated in terms of the following identity:

$$(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c) = a^3+b^3+c^3-3abc \tag{6}$$

This identity has appeared in earlier papers ([6, 9]) on the solution of cubic equations. It is very similar to Eq. (2), from which it can be derived by replacing a with a/ω. From Eq. (6), virtually the same steps presented above lead to the traditional form of Cardano's solution to the cubic. The symmetry of Eq. (2) may make its verification somewhat simpler than the verification of Eq. (6). Otherwise, either identity provides a simple approach to solving the cubic.

A streamlined version of the Cardano solution is particularly simple and memorable. Following the presentation in [10], begin with the cubic in the form

$$x^3 = px + q$$

and replace x with $b + c$. That leads directly to the equation

$$3bcx + b^3 + c^3 = px + q$$

Now match the coefficients of x : make $3bc = p$ and $b^3 + c^3 = q$. These conditions are virtually identical to those used earlier, and allow us to find b and c in terms of p and q. From this point on, the derivation is essentially the same as what was presented before.

There is a large literature on solving the cubic. References [3], [4], [9], and [11] are representative samples. Besides the previously cited presentation in [10], there is a recently published version in [5]. A translation of Cardano's published solution appears in [2]. Kleiner [7] provides an interesting discussion of the role of the solution of the cubic in the development of complex numbers. There is also an interesting history associated with Cardano's publication and his dispute with Tartaglia [1, 8].

References

[1] Carl Boyer. *A History of Mathematics,* 2nd edition, revised by Uta C. Merzbach, Wiley, New York, 1991, pp 282 – 286.

[2] Ronald Calinger, ed. *Classics of Mathematics,* Moore Publishing, Oak Park, Illinois, 1982, pp 235 – 237.

[3] H. B. Curtis. A derivation of Cardan's formula. *American Mathematical Monthly*, 51(1), January, 1944, p. 35.

[4] Orrin Frink Jr. A method for solving the cubic. *American Mathematical Monthly*, 32(3), March, 1925, p. 134.

[5] Jan Gullberg. *Mathematics from the birth of numbers,* W. W. Norton, New York, 1997, p. 318.

[6] Morton J. Hellman. A unifying technique for the solution of the quadratic, cubic, and quartic. *American Mathematical Monthly*, 65(4), April, 1959, pp. 274 – 276.

[7] Isreal Kleiner. Thinking the Unthinkable: The story of the complex numbers (with a moral). *Mathematics Teacher*, 81, October, 1988, pp. 583 – 592. Reprinted in Frank J. Swetz, ed., *From five fingers to infinity,* Open Court, Chicago and La Salle, Ill, 1994, pp. 711 – 720.

[8] Victor J. Katz. *A History of Mathematics - an Introduction,* Harper Collins, New York, 1993, pp 330 – 334.

[9] E. J. Oglesby. Note on the algebraic solution of the cubic. *American Mathematical Montly*, 30(6), Sept - Oct, 1923, pp. 321 – 323.

[10] John Stillwell. *Mathematics and its History,* Springer-Verlag, New York, 1989, pp 55.

[11] Robert Y. Suen. Roots of cubics via determinants. *College Mathematics Journal*, 25(2), March, 1994, pp. 115 – 117.