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Archimedes in the 5th Dimension

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Among the many discoveries of Archimedes, the reknowned Greek mathematician (ca 250 BC), one that is particularly striking relates the volumes of a sphere, cylinder and cone. This leads to our familiar formula for the volume of a sphere, $V = (4/3)\pi r^3$, and also revealed to Archimedes the surface area of the sphere. Indeed, Archimedes deduced that the ratio between the volume of a sphere and its circumscribing cylinder is the same as the ratio between the surface areas of these solids. So great was his admiration of this discovery that he had it inscribed on his tombstone.

Archimedes drew on mechanical principles (particularly the law of the lever) to make such discoveries, although he also constructed geometric proofs to substantiate his results. The argument relating the sphere, cone, and cylinder is formulated in terms of equilibria and balancing points. The heart of the argument can be understood a bit more easily in terms of simple geometry: The volume of a cylinder is equal to the sum of the volumes of a sphere and a cone because the same can be said about their cross-sectional slices. Interestingly, this idea can be extended very naturally to five dimensional figures, as we shall see below. First, though, we will take a more detailed look at the situation in our three familiar dimensions.

In three dimensions

Consider a hemisphere, cylinder, and cone, as shown in Figure 1.

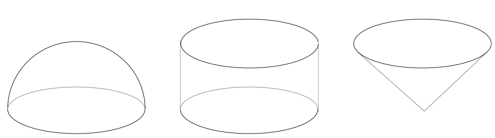

Figure 1. Sphere, Cylinder, Cone

The dimensions of the three figures line up. That is, they all have equal height r, the radius of the hemisphere, and this same r is the radius of the circular base of the cylinder and top of the cone. Note that this also implies that there is a 45 degree angle between the central axis of the cone and the lateral surface.

Now consider a cross sectional slice through all three figures, at a common height h, as in Figure 2.

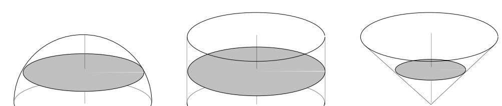

Figure 2. Slice at height *h*.

Each figure has a circular cross section. For the cylinder, the radius of the circle is r. For the cone, the radius is equal to h. And for the sphere, the radius of the slice is $\sqrt{r^2 - h^2}$ (see Figure 3).

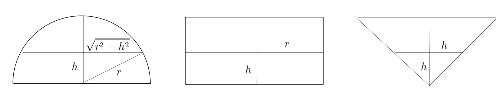

Figure 3. Side view.

Accordingly, the areas of these slices are, respectively, πr^2, πh^2, and $\pi(r^2 - h^2)$. We see at once that the area of the hemispherical slice plus the area of the conical slice equals the area of the cylindrical slice. Therefore, the volume of the hemisphere plus the volume of the cone equals the volume of the cylinder.

Now we can use known formulas for volumes of cylinders and cones to deduce the volume of the hemisphere. Since the cylinder has volume $\pi r^2 \cdot r = \pi r^3$, and since the cone has just one third of that volume, the hemisphere must have volume $V = (2/3)\pi r^3$. Of course, the full sphere is twice as large, giving the familiar $V = (4/3)\pi r^3$ for the volume of a sphere of radius r.

On to dimension 5

What does it mean to talk about volumes in five dimensions? First of all, we think of five dimensional space as made up of points (x, y, z, w, u) whose coordinates refer to positions along five mutually perpendicular axes. I have never actually *seen* five mutually perpendicular axes; to do so, I would have to exist in five dimensional space. So I have to use my imagination, and take the existence of these axes as a given.

In fact, there are a number of aspects of volumes that will just be assumed, in analogy with the geometry of three dimensions. One of these is the idea of distance or length. The distance from the origin to (x, y, z, w, u) is given by

$$\sqrt{x^2+y^2+z^2+w^2+u^2}.$$

This follows from the Pythagorean theorem, and some reasonable assumptions about perpendicularity, and can be found in any calculus or linear algebra book.

Continuing, a theory of volume in five dimensions can be developed in perfect analogy with volume in three dimensions. We define a unit cube to be a set of points something like this:

$$\begin{array}{ccc} 0 & \le & x & \le 1 \\ 0 & \le & y & \le 1 \\ 0 & \le & z & \le 1 \\ 0 & \le & w & \le 1 \\ 0 & \le & u & \le 1 \end{array}$$

and declare that it has a volume of 1. Visualize it as the region of space that has width 1 parallel to each of the five perpendicular axes. We can also think of this as the Cartesian product of five copies of the unit interval [0,1].

We assume that five dimensional volume is additive and translation invariant. That means that if you decompose a figure into pieces, the volume of the whole is the sum of the volumes of the pieces, and that congruent pieces have equal volume. From these assumptions, one deduces that the volume of a product of five intervals is the product of the lengths of the intervals. For example, a five dimensional rectangular prism with dimensions 1 by 2 by 3 by 4 by 5 would have volume $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Another important aspect of volumes is also related to Cartesian products. Suppose that (x, y, z) is free to roam throughout a three dimensional region R, and that (w, u) is free to roam about a plane region S. Then all possible points (x, y, z, w, u) with (x, y, z) in R and (w, u) in S populate a region in five dimensional space denoted by $R \times S$. The volume of that region is found by multiplying the volume of R by the volume of S. This is really the same idea as described above for the product of five intervals. It also leads to the familiar formulas for three dimensional volumes of cylinders (area of the base times the height). One has to be a little relaxed with the word *volume* however. If we are talking about a two dimensional object, volume really means area. For a one dimensional object volume is length. This collision of terminology inspires many writers to adopt a neutral term, *measure*, that stands for length, area, and volume depending on the dimensional context. But I will stick with *volume*, trusting you to understand when I really mean area or length.

At last, it is possible to describe the five dimensional version of Archimedes' observation linking the hemisphere, cone, and cylinder. In five dimensions, we consider four figures: a 5D-hemisphere, a 5D-cylinder, a 5D-cone, and a circle-cone. The circle-cone is actually a Cartesian product $R \times S$ where R is a two dimensional filled-in plane circle (hereafter known as a disk), and S is a three dimensional cone. We slice each of the four objects at a single height h along the u axis, and find that the volume of the slice of the 5D-hemisphere is equal to the volume of the slice of the 5D-cylinder + the slice of the 5D-cone – the slice of the circle-cone. Therefore, the five dimensional volume of the full figures made up of all such slices share the corresponding equation: hemisphere = cylinder + cone – circle-cone. As in the three dimensional case, this leads to a formula for the volume of the sphere.

For all of this to make sense, you need to have some idea what I mean by spheres, cones, cylinders, and slices in five dimensions. Let's begin with the sphere, made up of all the points at distance R or less from the origin. The point (x, y, z, w, u) is in the sphere if and only if $x^2 + y^2 + z^2 + w^2 + u^2 \le r^2$.

Similarly, a point (x, y, z, w) is in the 4-sphere of radius a if and only if $x^2 + y^2 + z^2 + w^2 \le a^2$. The four dimensional volume of this four-sphere is given by $V = (1/2)\pi^2 a^4$. This can be derived through an appropriate slicing analysis, but it will be too great a distraction to go into that in detail now.

Based on our understanding of the 5-sphere, what should we mean by a slice? In analogy with the three dimensional case, a slice perpedicular to the u axis is determined by assigning a fixed value to u. For example, set $u = 2$. The points of the sphere that have the form $(x, y, z, w, 2)$ make up one slice of the sphere. And such a point will satisfy $x^2 + y^2 + z^2 + w^2 \le r^2 – 4$.

Observe that this slice is a four dimensional object, in fact, a sphere of radius $\sqrt{r^2-4}$. If we slice at $u = h$, we obtain a four dimensional sphere of radius $\sqrt{r^2-h^2}$. Using the volume formula for the 4-sphere, we obtain $V = (1/2)\pi^2(r^2 - h^2)^2$ as the volume of one slice of the 5-sphere. Actually, we are only looking at slices for positive values of h, so we should really think of these as slices of a hemisphere.

Next consider a cylinder. In three dimensions, a cylinder is a Cartesian product of a disk with a perpendicular line segment. For example, consider the points (x, y, z) for which (x, y) is on or in the unit circle in the plane, and z lies between 0 and 1. That is precisely the description of a cylinder centered on the z axis with radius 1 and height 1. Each cross sectional slice is a disk of radius 1.

The five dimensional analog of this is a Cartesian product of a four dimensional sphere with an interval. For the case at hand, we allow (x, y, z, w) to be any point in the 4-sphere of radius r, and restrict u to lie between 0 and r. This defines our five dimensional cylinder, with equal radius and height r. Slicing this object at $u = h$ produces a 4-sphere of radius r. The four dimensional volume of the slice is $(1/2)\pi^2 r^4$.

The five dimensional cone is just a little more involved. Again begin with the three dimensional version. One way to characterize that cone is by describing its slices. We know that each slice is disk, and that radii of these disks increase linearly from 0 at the vertex to whatever radius we have specified for the top of the cone. If the height and top radius are both r, then the radius of each slice is equal to the height of that slice above the vertex. That is, the slice at height h will be a disk of radius h. Now we can extend this idea by analogy to five dimensions. We want the slice at $u = h$ to be a 4-sphere of radius h.

In symbols, $u = h$ implies $x^2 + y^2 + z^2 + w^2 \le h^2$. That leads to the inequality $x^2 + y^2 + z^2 + w^2 \le u^2$ as the definition of our cone. This agrees with other conceptualizations of the cone. For example, for any point P on the cone, the line from the origin to P makes a constant angle with the u axis. In any event, with this understanding of a cone, we see that the slice at $u = h$ is a 4-sphere of radius h. The four dimensional volume of this slice is $(1/2)\pi^2 h^4$.

Finally we come to the circle-cone. This is defined as a Cartesian product. Let (x, y) be any point of a disk of radius r. Thus, $x^2 + y^2 \le r$. Let (z, w, u) be any point of a cone centered on the u-axis with apex at the origin, and whose height and top radius each equal r. Then $z^2 + w^2 \le u^2$. The circle cone is made up of the points (x, y, z, w, u) satisfying both conditions.

Now what is the slice for u equal to a fixed value h? We still can have any (x, y) in the disk of radius r. But now we have (z, w, u) with $z^2 + w^2 \le h^2$ because $u = h$. This shows that the slice is a Cartesian product of two planar disks, one with radius r and the other with radius h. The four dimensional volume of the slice is therefore $\pi r^2 \cdot \pi h^2 = \pi^2 r^2 h^2$.

At this point, we are in a position to compare the corresponding slices of the hemisphere, cylinder, cone, and circle-cone. To simplify the comparison, the earlier results are combined in the following table:

Figure	Volume of Slice
Hemisphere	$\frac{1}{2}\pi^2(r^2 - h^2)^2$
Cylinder	$\frac{1}{2}\pi^2 r^4$
Cone	$\frac{1}{2}\pi^2 h^4$
Circle-Cone	$\pi^2 r^2 h^2$

It is evident that the volumes of these slices obey the Archimedes-like equation

Hemisphere = Cylinder + Cone – Circle-Cone.

Consequently, the five dimensional volumes of these solids must be related in the same way.

Now the volumes of the cylinder, cone, and circle-cone are easy to find. For the cylinder, volume is base times height, where the base is a 4D sphere of radius r, and the height is also r. This gives $V_{cyl} = (1/2)\pi^2 r^5$. Next, we need the volume of a cone in five dimensions. Recall that in three dimensions, the volume for a cone is 1/3 base × height. It turns out that in four dimensions the formula is 1/4 base × height and in five dimensions it is 1/5 base × height. So the volume of our cone is 1/5 the volume of the cylinder, and hence, $V_{\text{cone}} = (1/10)\pi^2 r^5$. As the circle-cone is a Cartesian product, it's volume is found by multiplying the volume of the three dimensional cone $(1/3)\pi r^3$ by the area of the circle (πr^2). Thus, $V_{cir\text{-cone}} = (1/3)\pi^2 r^5$.

Putting all of these results together, we find

$$\begin{aligned} V_{\text{hemisphere}} &= V_{\text{cyl}} + V_{\text{cone}} - V_{\text{cir-cone}} \\ &= \pi^2 r^5 \left(\frac{1}{2} + \frac{1}{10} - \frac{1}{3} \right) \\ &= \frac{4}{15}\pi^2 r^5 \end{aligned}$$

This gives the volume for five dimensional sphere as $V = (8/15)\pi^2 r^5$.

This is a result and a derivation that Archimedes would have appreciated. To be sure, a few of the missing details should be checked. These include the formula for the volume of a 4-sphere, as well as the pattern relating volumes of cylinders and cones in various dimensions. Both are readily verified using a slicing analysis and simple integration. You should be able to work these out on your own, now that you are familiar with higher dimensional slicing.

It is fun to see how much you can figure out about geometry in higher dimensions, and it is surprising how much you can work out in great detail. An interesting topic for further exploration is the volume of spheres in n dimensions. In fact, there are nice formulas for spheres in every dimension, but the even and odd dimensions follow different patterns. Here is one nice tidbit: for even dimension $2n$ the volume of the unit sphere is $\pi^n/n!$. That means if you add up the volumes of all the even dimensional unit spheres, you get e^π.

For more information on this topic, try doing an internet search on the volume of an n-sphere. Similarly, you can find many interesting references related to Archimedes and the volume of the sphere by searching for *Archimedes sphere cone cylinder*. For a more comprehensive and conventional source on the work of Archimedes, see Sherman Stein's 1999 MAA book *Archimedes: What Did He Do Besides Cry Eureka*?