Integrability and the glog Function

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New Elementary Functions? The *elementary functions* that are studied in calculus class are defined by long tradition: rational functions, trigonometric functions and their inverses, exponentials and logs. But we should not think that these are the only functions which deserve a place in the mathematics curriculum. For example, [2] makes a persuasive case for the Lambert W function, defined as the inverse of xe^x. There are also functions which were important in the history of mathematics, but which modern texts ignore. Who remembers the Gudermannian function?

Increasingly, computer algebra systems are playing a role in mathematics courses. These systems have an enormous capacity for storing information, and support a far richer library of named functions than what is found in textbooks. The W function just mentioned appears in both Maple and Mathematica (called `ProductLog` in the latter). And these computer systems do not keep their non-standard functions tucked away in some obscure corner known only to specialists. Simply enter an innocent looking integral or differential equation, and unfamiliar results can pop up without warning. In these cases, the answer provided by the computer may provoke a puzzled response. *What on earth is* `ProductLog`*?* Exploring the properties of these unfamiliar functions is a wonderful opportunity for both students and teachers.

This note reports such an exploration for a function called *glog* (pronounced *gee-log*). As reported in [3], glog was invented to solve a class of algebraic equations. It is defined as the inverse of e^x/x, making it a close cousin of W. Indeed,

$$W(x) = -\text{glog}(-1/x) \quad \text{and} \quad \text{glog}(x) = -W(-1/x) \tag{1}$$

so that glog and W can be used interchangably in a number of applications. Here, the emphasis will be on questions connected with symbolic integration, a context in which glog and W have different properties.

The basic question is this: when working with glog or W, which functions have closed form integrals? Using just the familiar *elementary functions* of calculus, the question of closed form integrability has been well studied; [4, 5] provide nice introductions to this subject. But what happens when a new function is introduced? On the simplest level, W itself has a closed form integral, but I have been unable to discover whether glog does. On the other hand, there are a number of simple forms involving glog with nice symbolic integrals. It is also natural to wonder whether the addition of these new functions enlarges the set of elementary functions which have closed form integrals. Unfortunately, I do not have an answer to this question, although I will offer some partial results connected with the integrability of glog.

Definitions and Notation. As mentioned, the glog function is defined as the inverse of e^x/x; W is the inverse of xe^x. It is a bit of an abuse of notation to refer to these as *functions* since each is double valued over a part of its domain, but that will present no difficulties in what follows. The *elementary functions* are as described above: rational functions, trigonometric functions and their inverses, e^x, $\log x$, and all combinations of these under the operations of arithmetic, as well as function composition. Similarly, the *g-elementary functions* are all the combinations of the elementary functions together with glog. This is a strictly larger set of functions because glog is known not to be elementary [4]. In light of (1), there is no point in also defining *W-elementary* functions, for these are just the same as the g-elementary functions.

If a function has an elementary function anti-derivative, it is said to be *elementary function integrable,* abbreviated *EFI.* If the antiderivative is g-elementary than the function is *g-elementary function integrable*, denoted *gEFI.* Since the elementary functions are a subset of the g-elementary functions, it is clear that every EFI function is also gEFI. Observe also that each EFI function is elementary, being the derivative of an elementary function. This observation depends on the fact that the elementary functions are closed under differentiation. A parallel argument shows that every gEFI function is g-elementary, reflecting the fact that the derivative of glog is g-elementary. More specifically,

$$\operatorname{glog}'(x) = \frac{\operatorname{glog}(x)}{x\operatorname{glog}(x) - x}$$

as is easily derived using implicit differentiation.

Now the questions posed above can be stated succinctly:

1. Which g-elementary functions are gEFI?
2. Are there any elementary functions that are gEFI but not EFI?

The additional question of whether any g-elementary functions are EFI is quickly disposed of: if a function is EFI then it is elementary, so the only g-elementary EFI functions are those that are actually elementary. In particular, glog itself is not EFI.

Is glog gEFI? I do not know. I have succeeded neither in integrating glog, nor in proving that it has no closed form integral. As a partial result, there is the following

Proposition: For any elementary function f, $f(\operatorname{glog}(x))$ is not an antiderivative of $\operatorname{glog}(x)$.

Proof: Suppose to the contrary that f is an elementary function and that

$$\frac{d}{dx} f(\mathrm{glog}(x)) = \mathrm{glog}(x).$$

This leads to

$$\begin{aligned} f'(\mathrm{glog}(x))\mathrm{glog}'(x) &= \mathrm{glog}(x) \\ f'(\mathrm{glog}(x))\frac{\mathrm{glog}(x)}{x\ \mathrm{glog}(x) - x} &= \mathrm{glog}(x) \\ f'(\mathrm{glog}(x)) &= x\ \mathrm{glog}(x) - x. \end{aligned}$$

Now replace x in this identity with e^y/y, to obtain

$$f'(y) = e^y - \frac{e^y}{y}$$

and hence

$$\frac{e^y}{y} = [e^y - f(y)]'.$$

This would imply that e^y/y is EFI, which is false [5]. This contradiction shows that no such function f can exist, and completes the proof.

A slight generalization of this result can be proved by similar methods: if $f(x, y)$ is an elementary function, then $f(x, \mathrm{glog}(x))$ cannot be an antiderivative of glog. The original version of the proposition says nothing about expressions such as x $\mathrm{glog}(x)$ and $\mathrm{glog}(x)/x$; the generalized version shows that such expressions cannot be antiderivatives for glog. Although these results do not show that glog cannot be gEFI, they make that conclusion seem quite plausible.

Some gEFI Functions. A simple example of a gEFI function is $\mathrm{glog}(x)/x$ because

$$\int \frac{\mathrm{glog}(x)}{x}\ dx = \frac{\mathrm{glog}^2(x)}{2} - \mathrm{glog}(x).$$

Notice that the anti-derivative of $\mathrm{glog}(x)/x$ is not an elementary function, for otherwise we could solve for glog and deduce that glog is elementary, a contradiction. So $\mathrm{glog}(x)/x$ is gEFI but not EFI.

Generalizing the preceding example, for $n \geq 1$

$$\int \frac{\mathrm{glog}^n(x)}{x}\ dx = \frac{\mathrm{glog}^{n+1}(x)}{n+1} - \frac{\mathrm{glog}^n(x)}{n}.$$

This result, in turn, is handy for integration by parts. Thus

$$\begin{aligned} \int \mathrm{glog}^n(x)\ dx &= \int x\frac{\mathrm{glog}^n(x)}{x}\ dx \\ &= x\left(\frac{\mathrm{glog}^{n+1}(x)}{n+1} - \frac{\mathrm{glog}^n(x)}{n}\right) - \int \frac{\mathrm{glog}^{n+1}(x)}{n+1} - \frac{\mathrm{glog}^n(x)}{n}\ dx. \end{aligned}$$

This equation can be rearranged to obtain the following recursion

$$\int \text{glog}^{n+1}(x)\ dx = \frac{x}{n}\left(n\text{glog}^{n+1}(x) - (n+1)\text{glog}^{n}(x)\right) - \frac{n^2-1}{n}\int \text{glog}^{n}(x)\ dx.$$

For $n = 1$, the integral on the right side drops out, to produce

$$\int \text{glog}^2(x)\ dx = x(\text{glog}^2(x) - 2\text{glog}(x)).$$

It is now easy to compute integrals for successive powers of glog, and the following pattern emerges: for $n \geq 2$

$$\int \text{glog}^n(x)\ dx = x\left[\text{glog}^n(x) + \sum_{k=1}^{n-1}(-1)^k \frac{n(n-2)!}{(n-k-1)!}\ \text{glog}^{n-k}(x)\right].$$

Although this formula has a certain kind of charm, and is straightforward to verify, a specific example does a much better job of conveying the pattern involved. With $n = 5$, we find

$$\int \text{glog}^5 x\ dx = x[\text{glog}^5 x - 5\text{glog}^4 x + 5\cdot 3\text{glog}^3 x - 5\cdot 3\cdot 2\text{glog}^2 x + 5\cdot 3\cdot 2\cdot 1\text{glog}\ x.]$$

All of these examples involve gEFI functions, and so they shed no light on the existence of elementary gEFI non-EFI functions.

gEFI non-EFI Functions. The interest in gEFI non-EFI functions is prompted by a simple question: does the introduction of glog enlarge the set of elementary functions which are integrable in closed form? An affirmative answer can be established by one example of an elementary function can be integrated using glog, but not otherwise.

I do not have an example of such a function, but here is a related, if somewhat foolish, example:

$$\int \frac{1}{x}\ dx = \text{glog}(x) - \log(\text{glog}(x)).$$

The example is foolish because we already know how to integrate $1/x$, and the example is simply a restatement of the identity

$$\log(x) = \text{glog}(x) - \log(\text{glog}(x)).$$

However, the example does show that elementary functions can have integrals expressed in terms of glog.

A better example would be to integrate something like e^x/x using glog. Clearly e^x/x is an elementary function, yet, as mentioned earlier, it is not EFI. On the other hand, as suggested by Figure 1,

$$\int_1^t \frac{e^y}{y}\ dy = e^t - e - \int_e^{e^t/t} \text{glog}(x)\ dx.$$

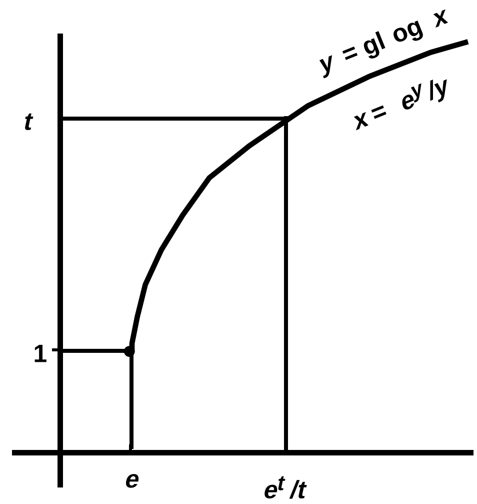

Figure 1: Integrals of glog(x) and e^y/y

Accordingly, showing that glog is gEFI would also show that e^x/x is gEFI.

What if glog is not integrable? A variation on the argument above might still lead to a gEFI, non-EFI function. The relation between the integrals of e^x/x and glog(x) occurs in a similar way for any pair of inverse functions. The idea would be to find a pair of inverses of which one is elementary but not EFI, and the other one is gEFI. Candidate pairs are easily formulated. Some examples are

$$y = \frac{x}{\log(x)} \quad \Leftrightarrow \quad x = y\ \mathrm{glog}(y)$$
$$y = \frac{\log(x)}{x} \quad \Leftrightarrow \quad x = \frac{1}{y}\mathrm{glog}\left(\frac{1}{y}\right)$$
$$y = \frac{x}{e^x} \quad \Leftrightarrow \quad x = \mathrm{glog}\left(\frac{1}{y}\right)$$

None of these pairs leads to a gEFI non-EFI function. In each case, either the elementary function is EFI, or I have been unable to show that the inverse is gEFI. Rather than beginning with inverse function pairs, an alternative is to start with a nice gEFI function and hope that its inverse is elementary, and not EFI. For example, let $h(x) = x^2/(x-1)$. Then, as shown earlier, $h(\mathrm{glog}(x))$ is gEFI, with integral $x\mathrm{glog}(x)$. Now the inverse function is $e^{h^{-1}(x)}/h^{-1}(x)$, and is elementary since h^{-1} is. This illustrates the idea of beginning with a gEFI and checking that the inverse is elementary. Unfortunately, for this particular example, the inverse function is also EFI: its integral becomes, with the change of variables $x = h(u)$,

$$\int \frac{e^u}{u} h'(u)\ du = \int \frac{u-2}{(u-1)^2} e^u\ du = \frac{e^u}{u-1}$$

At this point, the strategy of finding a pair of inverse functions, one of which is elementary and not EFI, the other of which is gEFI, remains no better than a promising lead. Whether it can be used to find an elementary gEFI function that is not EFI remains to be seen.

Differential Equations. There is an obvious direction for generalizing these questions about integration using glog. Integration amounts to solving a very special kind of differential equation. How might glog be used to solve other kinds of differential equations? For example, how would you solve this one?

$$yy' = \frac{y+1}{x}$$

Of course you *could* separate the variables and integrate both sides. But you could also make the substitution $u = y + 1$ to obtain

$$u' = \frac{u}{x(u-1)}$$

and recognize that $u = \text{glog}(x)$ is a solution. In fact, the general solution is $u = \text{glog}(cx)$. Another example is

$$(xy - x^2)y' = y^2$$

The solution is $y = x\text{glog}(cx)$, as the reader is invited to verify. It is also possible to solve this equation by separating variables after making the substitution $y = xv$. Are there differential equations that can be solved using glog that cannot be solved by some other means? That remains an open question.

The W Function. So far, almost nothing has been said about W. In contrast to glog, the W function does have a simple integral: $x(W(x) - 1 + 1/W(x))$ ([2]). However, that does not mean that W necessarily enlarges the class of integrable elementary functions. The analogy with glog breaks down because the inverse of W, xe^x is itself integrable. On the other hand, W does permit the integration of a number of differential equations, accounting for several of the applications of W cited in [2].

Conclusion. Introducing glog (or equivalently, W) leads in a natural way to questions about integration in closed form. The examples above demonstrate that glog gives rise to a number of functions with simple integrals. However, a fundamental question, whether glog itself has a simple integral, remains unresolved. On one hand, if glog *is* integrable in closed form, it provides an immediate example of a gEFI non-EFI function, and shows that glog extends the set of integrable elementary functions. On the other hand, the partial results proven above make it seem unlikely that glog is integrable in closed form, and suggest that some other example of a gEFI non-EFI function should be sought. For now, these will have to remain open questions.

References

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