"In the case of a third degree polynomial, that is, in the case of a cubic, Marden's theorem tells us *exactly* where to find the roots of f'."

The Most Marvelous Theorem in Mathematics

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What makes a theorem great? There are many factors that contribute: generality, utility, power, symmetry. But for me, a great theorem is one that surprises. If you have ever read the statement of a theorem that made you exclaim "No Way!" then you know what I mean.

This article is about my candidate for the Most Marvelous Theorem in Mathematics. It is simple to understand, answers a natural question, and involves familiar mathematical concepts, like derivatives and polynomials. And if it doesn't surprise you, and here I mean slap-yourself-in-the-forehead and swallow-your-gum level surprise, you are way more insightful than I am. It does require a little bit of background knowledge, though, mainly about complex numbers and ellipses, so before stating this marvelous theorem, let me give a quick review of those topics.

Complex Numbers

The complex numbers are formed by combining the square root of negative 1, customarily denoted i, with the reals. Every complex number can be written in the form $a + bi$ with a and b real numbers. Addition, subtraction, and multiplication are performed by treating i as if it were a variable, except that every appearance of i^2 is replaced by –1. For example, $(3+2i)(3–5i)$ is computed in the same way as $(3+2x)(3–5x)$. (Repeat after me: firsts, outers, inners, lasts.) That gives us $9 – 15i + 6i – 10i^2$. But we know that i^2 equals –1, so the answer simplifies to $19 – 9i$. Division of complex numbers is also possible, but won't be needed in this discussion.

Although we treat i like a variable in some ways, you should think of the complex numbers as a number system made up of constant numbers, sort of like the real numbers. So $3+4i$ is a specific number, just the way 5/11, π, and $\sqrt{3}$ are. However, whereas the real numbers are visualized as points on a line, complex numbers are visualized as points in a plane. For example, we think of $3+4i$ as sitting at the point (3,4) in the complex plane. In this way, sets of complex numbers are associated with figures in the plane, just as sets of real numbers are associated with subsets of the real line. To carry this idea a little further, any two *real* numbers determine an interval on the *real line*, while any three *complex* numbers determine a triangle in the *complex plane*.

Complex numbers can be used as constants to define functions. Here is a simple example: $f(z) = z^2 + (3+2i)z + (5 – 4i)$. This is a quadratic polynomial, and the coefficients are complex (rather than real) constants. The letter z is a traditional choice for a complex variable, and it makes perfect sense to substitute a complex number for z in our function. Say we take $z = 1+i$. Then substitution leads to $f(1+i) = (1+i)^2 + (3+2i)(1+i) + (5+4i)$. With a little work, you can simplify this back to the final $a + bi$ form, arriving at $6+11i$. In summary, $f(1+i) = 6+11i$. This illustrates the operation of a complex function: we put in a complex number $(1+i)$ for the variable, and we get out a complex number $(6+11i)$ for the value of $f(z)$. Note that we can consider for complex functions many of the same questions that arise with real functions. We might ask for the roots of $f(z)$—meaning complex values of z that make $f(z)$ come out to 0. Or we might compute the derivative of $f(z)$, as $f'(z) = 2z+(3+2i)$. In the theorem to be presented, functions of this sort play a prominent role.

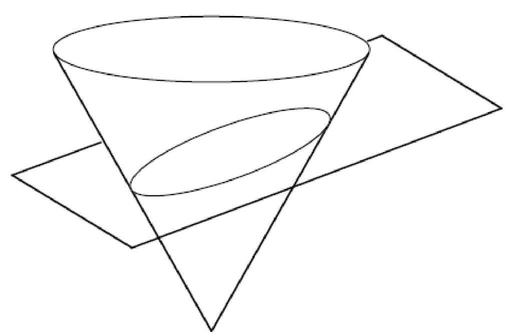

Figure 1. Cutting a Cone at an Angle.

Ellipses

In what is seemingly a completely unrelated direction, we must also review ellipses. Conceptually, an ellipse is the oval shaped curve that results when a plane is put through a cone at an angle (though not too steep an angle, because we insist on a *closed* curve). (See Figure 1.) An ellipse has two special interior points, called the *foci* (pronounced *foe-sigh*; plural of *focus*). The lore known about ellipses and their foci fills volumes, but here we will be content to remember two facts. First, each focus is literally named, in the optical sense. Thus, if an ellipse reflects light like a mirror, and if a point source is placed at one focus, all of the rays of light will reflect off the ellipse and reconverge at the other focus. (Figure 2.) Moreover, each ray of light will travel exactly the same distance. That is, from the first focus, to any point on the ellipse, and then to the other focus, the distance is always the same.

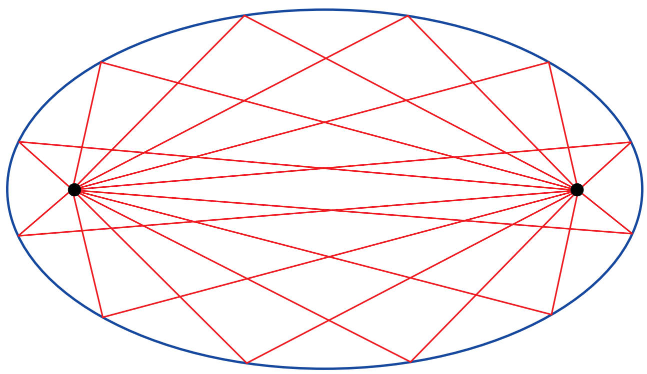

Figure 2. Foci of an Ellipse.

Marden's Theorem

Now I can state what I call Marden's Theorem, the most marvelous theorem in mathematics. Actually, I call it Marden's Theorem because I first read it in a book by Marden, but Marden himself attributed the theorem to Siebeck. In any case, the theorem concerns a relationship between the roots of a polynomial and the roots of its derivative. In the familiar setting of calculus, you have probably seen this kind of relationship in Rolle's Theorem. This states that when $f(x)$ is a differentiable function, in between any two roots of f there must be at least one root of f'. So if you know where the roots of f are, you get some idea about where roots of f' should be.

In the complex setting, there is a closely related theorem due to Lucas. Remember now that the roots of $f(z)$ are points in the plane, as are the roots of $f'(z)$. And what Lucas's Theorem says is this: if $f(z)$ is a polynomial, then surround the roots of f with the smallest possible convex polygon; that polygon will contain all of the roots of f'.

In the case of a third degree polynomial, that is, in the case of a cubic, Marden's theorem tells us *exactly* where to find the roots of f'. To begin with, a cubic will have three roots in the complex plane. We assume that those three roots are not all on a line, so that they form the vertices of a triangle. Now inside that triangle, inscribe an ellipse. This can be done in many ways, but if we insist that the ellipse be tangent to the sides of the triangle at their midpoints, the inscribed ellipse is unique. Like all ellipses, this inscribed ellipse has two foci. And those foci are the roots of f'.

The figure below illustrates the situation. The triangle has its vertices at the roots of f. The inscribed ellipse is tangent to the sides of the triangle at their midpoints. The marked points in the interior of the ellipse are its foci. And they are also the roots of the derivative of f.

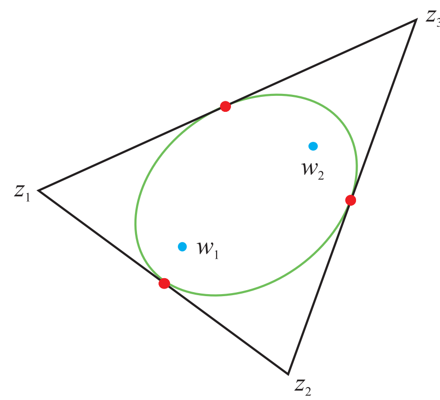

Figure 3. Marden's Theorem Illustrated: If z_1, z_2, z_3 are the roots of f, then w_1 and w_2, the foci of the inscribed ellipse, are the roots of f'.

Marden's Theorem can be stated succinctly as follows: If $f(z)$ is a cubic polynomial with complex coefficients, and if the roots of f are three distinct non-collinear points A, B, C in the complex plane, then the roots of f' are the foci of the unique ellipse inscribed in triangle ABC and tangent to the sides at their midpoints.

Isn't that amazing? Does it compel you to ask why? How can it be that the connection between a polynomal and its derivative is somehow mirrored perfectly by the connection between an ellipse and its foci? Seeing this result for the first time is like watching a magician pull a rabbit out of a hat. After thinking about it off and on for about thirty years, I still have that reaction. That is why this is my favorite theorem.

To learn more about this amazing theorem, and to see a proof, visit an interactive online presentation: "The Most Marvelous Theorem in Mathematics" by Dan Kalman that appears in the MAA's *Journal of Online Mathematics and Applications* (JOMA), `www.JOMA.org`. Alternatively, for a more traditional mathematical presentation see "An Elementary Proof of Marden's Theorem" by Dan Kalman, *American Mathematical Monthly*, Volume 115, Number 4, April 2008, pp 330-338.

So what do you consider to be the most marvelous theorem in mathematics? Send the editors your favorite, with a brief reason for your choice, and it might appear in a future issue of *Math Horizons*.