Teaching Linear Algebra: Issues and Resources[*]

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In recent years there has been much lively debate and creative discussion about improving the teaching of linear algebra. A special issue of this Journal [**7**] and a separately issued anthology [**6**] outline developments, and present a variety of viewpoints and issues related to teaching linear algebra, along with rich bibliographic resources for additional study. The resources contain material on goals of instruction, material to cover, methods of instruction, instructional technology, levels of abstraction and rigor, applications, student diversity, connections with other courses, and more.

In the summer of 1998, these matters were the focus of the Undergraduate Faculty Program (UFP) at the three week Park City Mathematics Institute (PCMI). (PCMI annually brings together a broad spectrum of mathematics researchers, teachers, and students in a collection of coordinated programs organized around a central theme. PCMI is an unusual and remarkable institution, and not sufficiently well known. It provides opportunities for interaction between segments of the mathematics community which depend heavily on each other but rarely communicate in depth. The PCMI website is at http://www.admin.ias.edu/ma/.) The purpose of this article is to make available to a wider audience some of the materials produced by the PCMI-UFP participants and resources we found useful. We will also discuss some of the experiences we had and insights we gained there which we found most new and interesting.

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PCMI-UFP Reports

Thirteen faculty participated in the 1998 PCMI-UFP, contributing to detailed reports on several different aspects of teaching linear algebra. We will begin by giving brief descriptions of these reports, which are available at the PCMI-UFP website http://pcmi.knox.edu/.

Teaching Linear Algebra: What Are the Questions? This paper (denoted TLAQ for future reference) covers a broad range of questions about the teaching of linear algebra, emphasizing issues that were discussed at the PCMI-UFP program.

Report on Technology. This report offers several resources for linear algebra teachers interested in possible uses of technology. The first section is a general overview of the potential benefits and pitfalls of technology in teaching mathematics. That is followed by a collection of several class projects illustrating uses of technology that have been found effective by the participants, and a small list of references to additional technology resources. A separate section provides instructions on how to use several technology products, with a comparison of three software packages and two graphing calculators.

Curriculum and Pedagogy Report. This reports on a teaching methods and learning theories seminar, centered around a series of readings. The UFP seminar members were joined at various times by distinguished visitors to PCMI, as well as by groups of mathematics education researchers and secondary teachers participating in other PCMI programs. Discussion topics included course content, articulation with secondary mathematics, effective methods of instruction and assessment, understanding how students learn, and ways to assess meaningful learning.

Textbook Reviews. The PCMI participants had several discussions about different models for linear algebra courses, and looked in detail at textbooks using a variety of approaches. The report has reviews of six textbooks.

Insights and Resources

Many of the topics considered during the PCMI-UFP would be familiar to readers following the literature on linear algebra instruction. However, in several areas we encountered less familiar topics or resources that we feel deserve consideration. These will be presented briefly in the following sections. More developed discussions of these topics can be found in reports at the PCMI-UFP website mentioned above.

Curriculum Issues. A primary starting point for a discussion of topics to include in a linear algebra course is the 1990 core curriculum of the Linear Algebra Curriculum Study Group [**5**]. Other references that make recommendations regarding content are [**9, 11**]. An article by Axler [**1**] argues for the exclusion of determinants from linear algebra, and presents an elegant mathematical development of eigenvalues and eigenvectors that makes no use of determinants. A related article by McWorter and Meyers [**21**] provides simple algorithms for hand computation of eigenvalues and eigenvectors, again without determinants. These articles are interesting if only because the viewpoints they take are so atypical.

The Linear Algebra Study Group curriculum recommendations, and most current textbooks, emphasize the applications of linear algebra as a motivation for study. A different approach is to treat linear algebra as a unifying framework for understanding mathematical structure. Such an approach is being developed by Barker and Howe [**2**], in the context of continuous symmetry groups. Lie theory is another context in which linear algebra finds a natural home. An overview of Lie theory, and its connections with undergraduate topics, is provided by Howe's paper [**15**].

Connections with Secondary Mathematics. Harel [**11**] has pointed out that while high school curricula provide several years of preparation for the ideas of calculus, concepts of linear algebra receive little mention. He proposes exposing students to some linear algebra methods and concepts at the secondary level, as a foundation for the abstraction and precision of college linear algebra courses.

Prompted by discussions with Harel, we became interested in the articulation between college and secondary mathematics. From secondary teachers present at PCMI, we learned that many high school curricula already include a number of topics from linear algebra, such as matrix representation of linear systems and matrix operations, but they are often covered briefly, if at all. The teachers explained that the scope and emphasis of secondary mathematics is strongly shaped by the advanced placement (AP) examinations in calculus. For able high school students, completing calculus has become the central focus. As long as topics from geometry and linear algebra do not appear on the AP examinations, parents and administrators will have little sympathy for spending much time on them. The secondary teachers also pointed out that among all the students enrolled in advanced secondary mathematics courses, few will eventually take a college linear algebra course, so it is difficult to justify including much linear algebra background material. But many students will study functions of several variables in calculus, so there is more reason for bringing back the study of three dimensional geometry in high school.

Our discussions with the secondary teachers convinced us that college mathematics faculty should be more informed and concerned about the curriculum of the secondary schools. One good starting point is the Third International Mathematics and Science Study (TIMMS). Information about TIMSS can be found at http://nces.ed.gov/TIMSS. See also [**22**].

Research on How Students Learn. Of all of the subjects we studied at PCMI, the issue of understanding how students learn was the one that made the greatest impact on us, leaving us doubtful that anyone really understands what mathematical concepts students know or how they learned them. Let us qualify that. We believe that it is possible to tell if students have mastered a specific topic at a specific time. But we doubt if anyone has a clear understanding of how students accomplish that, of the intermediate stages of understanding they may pass through along the way, of what their teachers did or did not do that assisted or impeded that learning, or of what understanding they will retain. From the exams we give in our own linear algebra courses, we know that many students do not master particular topics and have serious misunderstandings about many concepts. It should clearly interest any linear algebra teacher to discover why students find this material difficult, and how to help them most effectively.

There is not a great deal of published literature on how students learn linear algebra, although Harel [**11, 12, 13**] has been studying some aspects of this for several years. His papers provide some suggestions for linear algebra teachers. Carlson [**4**] presents an interesting hypothesis about the special difficulties that linear algebra presents for students, and Dubinsky [**9**] offers another point of view.

Research with Young Children. There has been much more research on how young children learn mathematics. Although this research is not directly applicable to the linear algebra classroom, it provides suggestive clues. We learned about some of the work in this area at PCMI, both in informal discussions with the mathematics education researchers, and in lectures presented to the entire PCMI assembly.

One such presentation was made by Michael Battista [**3**]. He began with a filmed interview of a young girl, perhaps seven years old. She was shown a diagram something like Figure 1 and asked to predict how many square tiles would be needed to cover the rectangle.

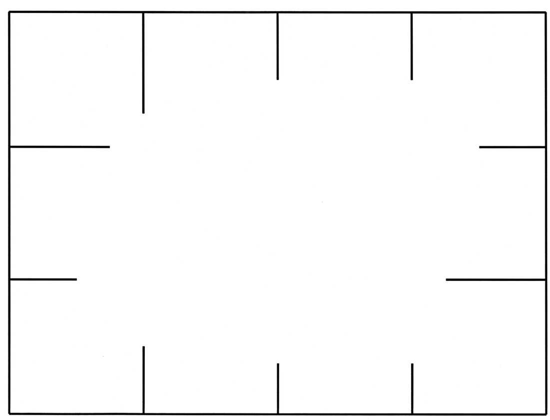

Figure 1. Tiling Exercise

In answer, she counted around the perimeter of the diagram, and then added a few more tiles to fill in the center. It was clear that she did not have a mental image of a rectangular array with a certain number of tiles in each row, and in each column. When she was given physical tiles, she successfully covered the diagram and counted out the twelve required tiles. Then the tiles were put away, and the child was again asked the original question. She reverted to the perimeter strategy, and when she came to estimate the number of missing tiles from the center of the array, she was no more able to correctly envision the tiles than she had been at the start.

That presentation left us with two conclusions. First, we realized that something as concrete as the organization of a set of tiles into an array is in fact an abstract construction. To master this concept a child needs a certain mental maturity as well as practice and repeated exposure. It was not obvious or natural for the child to conceptualize the arrangement of tiles as an array, even after she built the array herself out of physical tiles. Second, we saw that there is more to learning mathematics than we sometimes imagine. It is naive to think that teaching consists of showing students clearly what you want them to know. Such a view ignores the cognitive development that can be necessary even for so simple an idea as arranging tiles into rectangular arrays. Although the girl in Battista's study actually saw for herself how to arrange real tiles in an array, that did not make array arrangements natural or obvious to her. It is hard to imagine that telling her about arrays would have produced significantly different results.

What does this tell us about teaching linear algebra? By now, most teachers have heard the assertion that students have to *construct their own knowledge* in order to learn, but we are not sure how to assist students in doing this. It seems apparent that simply showing what is true and telling students what we wish them to know is not generally sufficient. Several studies of calculus have shown that students do not generally have a rich conceptual understanding of graphs and functions. These students are repeatedly shown how graphs and functions are related, but like the girl in Battista's study, they do not find the concepts obvious or natural. We in the PCMI-UFP agreed that linearity and independence are also concepts with which most students struggle, and it seems clear that simply doing a better job of telling and showing may not significantly improve their learning of such difficult topics.

What *can* we do to enable students' learning? We asked the mathematics education researchers at PCMI for advice. The essence of their responses was that we should become better listeners. Our discussions with students should be aimed not only at correcting misconceptions, but at understanding what the misconceptions are and how they develop, as well. To do this in an organized fashion, you might ask for a few volunteers willing to participate in interviews throughout a course. The point of the interviews is to obtain a detailed understanding how students think about linear algebra. Over time, interviews may reveal some general patterns in the thinking and misconceptions of our students that lead us to a better understanding of how they learn, although any such insights may well take several years to emerge. But in the short term, careful observation of students has value as an end in itself, in helping them see their misconceptions, and helping us to remember that the abstractions of linear algebra are *not* trivial and *do* require time and effort to acquire.

Ideas from Physics Education. During our consideration of how students learn, we also discussed some very interesting work which has been going on in the physics community during the past decade. Assessment tools have been developed to measure the qualitative understanding of basic principles retained by students after they have studied college physics [**14, 16, 23**]. They seem to show that even physics majors from prestigious schools fail to apply accepted principles of Newtonian mechanics to everyday events, applying instead naive understandings of motion and dynamics that physics has shown to be incorrect.

This has already led to changes in the way physics is being taught. One interesting case is a method developed by Eric Mazur (see for example [**20**]; also visit the website http://galileo.harvard.edu, particularly under the heading of *Peer Instruction*). Mazur teaches basic physics at Harvard, in a large lecture hall, and uses the following quick but effective collaborative learning method. After he talks for about fifteen minutes on a new topic, he gives a multiple choice question to find out if students have understood. Their answers are quickly tallied and reported. Often there is substantial disagreement about what is the correct answer, so he asks them to discuss the question for two or three minutes and then vote again. Usually, after lively discussions, most of the students vote correctly the second time. When this happens, Mazur affirms the correct result and goes on. Otherwise he discusses the students' misconceptions briefly. His students rarely miss class, for they have found out that these *put it together* questions are a key to passing the course!

Inspired by Mazur's approach, a subgroup of the PCMI-UFP participants made some preliminary steps toward developing questions that might assess conceptual learning in linear algebra. The results are contained in the Curriculum and Pedagogy Report posted at the PCMI-UFP website. TLAQ also contains some discussion of our experiences using an adaptation of Mazur's approach in linear algebra classes.

For us, participating at PCMI inspired a rededication to helping students achieve meaningful learning, and a renewed appreciation of how difficult this can be. As mathematicians, we are aware of the rich interconnections of different ideas and concepts. We would like our students to learn how different ideas fit together, supporting and validating each other, and contributing to a deeper understanding of each part. We know that understanding does not result from being told of each individual fact and principle, but comes from actively exploring a mathematical topic, discovering and rediscovering the interconnections until they become familiar and commonplace. But we who have developed understanding on this level risk forgetting the effort that came before: the missteps, false generalizations, incomplete and inconsistent conceptions. In full possession of the facts, we recite them piecemeal to our students as if we expect each new revelation to fit naturally and effortlessly into the pattern of what has already been presented. Teachers should be wary of this view, and skeptical of its effectiveness in the classroom. Meaningful learning is difficult to achieve and it rarely occurs unless students actively grapple with the ideas.

Technology. The software package MATLAB is one of the most natural for use in linear algebra, as it was developed specifically for matrix operations. Computer files for use with MATLAB are available from some textbook publishers (e.g., for use with Lay's text [**18**]), and specially designed MATLAB and MAPLE teaching codes developed at MIT are available at web.mit.edu/18.06/www/. MATLAB materials for teaching linear algebra have also been developed by a project called ATLAST. Funded by the NSF, the ATLAST Project presented a series of workshops in 1992–97 to acquaint linear algebra teachers with uses of instructional technology. Some of the materials developed at the workshops have been published [**19**]. There is a website for the project at www.umassd.edu/SpecialPrograms/Atlast/welcome.html.

There are also textbooks that have been developed for use with other software packages, and which rely on computer activities as the primary means of instruction. Examples are [**24**] and [**8**].

Another approach to instructional technology exposes students to mathematics within the familiar interactive medium of windows and webbrowsers, with the goal of making the interaction so natural that students can pose questions and seek answers with virtually no translation into a computer syntax. These sorts of activities are becoming increasingly common as *Java Applets* on webpages. The MAA's new *Journal of Online Mathematics* will feature a collection of such activities on its "Mathlets" page (www.joma.org/mathlet.html). See also archives.math.utk.edu/topics/linearAlgebra/html for examples related to linear algebra and the electronic column *Cut the Knot*, a regular feature of MAA Online (www.maa.org).

In a similar vein are webpage-like interactive environments that operate with software called Mathwright. The Mathwright Library website (www.mathwright.com) makes the software and a collection of activities freely available. The activities in the library were developed by teachers using a powerful development tool called Mathwright Author (see reviews [**10, 17**]). The Mathwright Library provides activities on a broad range of mathematical subjects, including a few appropriate for use in a linear algebra course, as well as many others that can give linear algebra teachers samples of the interactions possible with current technology.

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A Fate Worse Than Drugs

Edwin Rosenberg (Western Connecticut State University, rosenbergx@wcsub.ctstate.edu) found an advertisement that appeared, among other places, in the March 2001 issue of *Teen* magazine (page 29) objectionable, or at least ill-advised. It was not an advertisement for a product, but one against drugs, evidently placed by the "Office of National Drug Control Policy/Partnership for a Drug-Free America." There is a picture of a teen-aged female, and the text begins

> He's superfly. He wants your digits. But first he wants you to get high. Here's what you say.
>
> No way. I would rather go to math camp than smoke a joint.

For those not in touch with teen-aged slang, "your digits" means "your telephone number."

