Families of Curves and Their Envelopes

A family of curves is given by an equation equivalent to the form

$$F(x, y, p) = 0 \tag{1}$$

where for each value of p the equation defines a curve C_p in x and y. For example, the equation

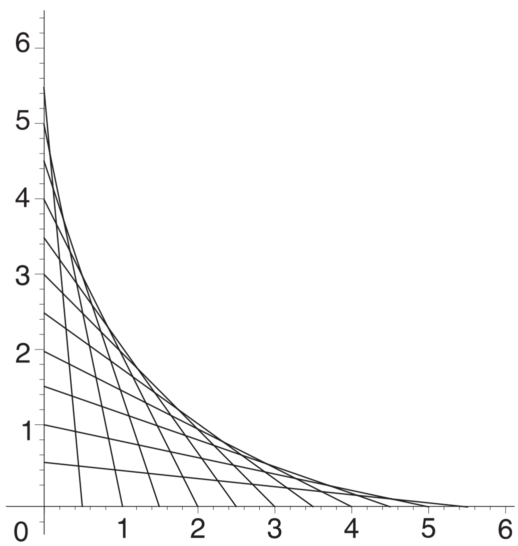
$$\frac{x}{p} + \frac{y}{6 - p} = 1 \tag{2}$$

defines a family of straight lines. For each p, we get the line in the xy plane whose x and y intercepts are p and $6 - p$, respectively. More generally

$$\frac{x}{a(p)} + \frac{y}{b(p)} = 1 \tag{3}$$

defines a family of lines C_p whose intercepts are $a(p)$ and $b(p)$, respectively.

The figure below shows several members of the family of lines defined by (2). The pattern of lines seems to create an additional curve, in this case marking a boundary of the region swept out by the lines. That curve, which is tangent at each point to one of the lines is called an *envelope* for the family of lines. Two methods for finding the envelope of a family of curves are given below.



The First Envelope Algorithm

To find the envelope of the family of curves (1), differentiate with respect to p. Then, from the two equations

$$\begin{aligned} F(x,y,p) &= 0 \\ \frac{\partial F}{\partial p}(x,y,p) &= 0, \end{aligned}$$

eliminate p to obtain an equation in x and y. It is the equation of the envelope.

Let us apply this algorithm to the family of lines (2). To be consistent with the general description of the algorithm, put the equation into the form

$$\frac{x}{p} + \frac{y}{6-p} - 1 = 0.$$

But since we will be differentiating with respect to p, let us we rewrite it as

$$(6-p)x + py - p(6-p) = 0. \tag{4}$$

Next, differentiate with respect to p, finding

$$-x + y - 6 + 2p = 0.$$

Can we eliminate p from these equations? Certainly. From the second

$$2p = x - y + 6.$$

Next, multiply both sides of (4) by 4, distributed as follows:

$$2(12-2p)x + 2(2p)y - 2p(12-2p) = 0.$$

Now substitute $x - y + 6$ for $2p$ and simplify:

$$\begin{aligned} 2(12-x+y-6)x + 2(x-y+6)y - (x-y+6)(12-x+y-6) &= 0 \\ 12x - 2x^2 + 2xy + 2xy - 2y^2 + 12y + x^2 - 2xy + y^2 - 36 &= 0 \\ -x^2 + 2xy - y^2 + 12x + 12y - 36 &= 0 \\ x^2 - 2xy + y^2 - 12x - 12y + 36 &= 0. \end{aligned}$$

This is the equation for the envelope. To verify that the equation is correct, add it to the graph of the family of lines using the applet.

The Second Envelope Algorithm

An alternative way to specify a family of curves is by describing each member parametrically. Consider a pair of equations

$$\begin{aligned} x &= f(t,p) \\ y &= g(t,p). \end{aligned}$$

Holding p fixed, these define x and y as functions of t, providing a parametric representation of one curve. Different values of p give different curves in the family.

In this setting, a point (x, y) is on the envelope curve if

$$\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial p} \end{vmatrix} = 0.$$

That gives us one equation in p and t. Use it to eliminate t from the parametric equations defining the family. That is, if the determinant equation gives us $t = t(p)$, then the parametric equations

$$\begin{aligned} x &= f(t(p),p) \\ y &= g(t(p),p) \end{aligned}$$

describe the envelope for the family of curves.

We can apply this to the general form of (3). For each p we have a line with intercepts $a(p)$ and $b(p)$. It can be expressed parametrically as

$$\begin{aligned} x &= (1-t)a(p) \\ y &= tb(p). \end{aligned}$$

Applying the second envelope method, set

$$\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial p} \end{vmatrix} = \begin{vmatrix} -a(p) & (1-t)a'(p) \\ b(p) & tb'(p) \end{vmatrix} = 0.$$

Suppressing the dependence of a and b on p, we obtain

$$ab't + ba'(1-t) = 0,$$

which gives

$$t = \frac{ba'}{ba' - ab'}.$$

We also see that

$$1 - t = -\frac{ab'}{ba' - ab'}.$$

Therefore, substitution in the parametric equations for our family of lines gives

$$\begin{aligned} x &= -\frac{a^2 b'}{ba' - ab'} \\ y &= \frac{b^2 a'}{ba' - ab'} \end{aligned}$$

defining the envelope curve parametrically in terms of p.

In the example of (2), we have $a(p) = p$ and $b(p) = 6 - p$. The envelope curve is thus given by

$$\begin{aligned} x &= \frac{p^2}{6} \\ y &= \frac{(6-p)^2}{6}. \end{aligned}$$

To verify that these equations are correct, enter them in the appropriate place in the envelopes applet, and examine the resulting graph.