Exercises for Uncommon Mathematical Excursions

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UME Problems

Chapter 1.

1. Reversing the order of coefficients for a polynomial gives the reverse polynomial. If $p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$ then the reverse polynomial is $-3x^4 + 7x^3 + 6x^2 - 11x + 5$.

   a. Prove that if $p(x)$ is a polynomial of degree n, then the reverse polynomial is $x^n p(1/x)$.

   b. Use reverse polynomials and Horner form to develop a strategy for mentally computing $p(x)$ when x is a unit fraction such as $1/2, 1/3$, or $1/5$. Apply your strategy to find $p(1/3)$ for the polynomial $p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$.

2. Converting binary to decimal form. Use Horner evaluation to convert the binary numeral 1101001 into decimal. Formulate a general algorithm for such conversions.

3. Reverse the process of the preceding problem to develop a simple conversion algorithm from decimal to binary representation.

4. On a calculator, compute 1.005^{36} using only the x^2 and multiplication operations. Hint: first express 36 in binary form. Note that this is the sort of computation needed to analyze compound interest questions. For example, 1.005^{36} is the accrual factor for three years of interest compounded monthly at .5% per month.

5. Differentiate using synthetic division, as on page 10 of UME, to find $p'(x)$ when $p(x) = 5x^3 + 6x^2 - 2x + 7$.

6. Use the method shown on page 23 of UME to approximate the cube root of 1.23 using just the squareroot operation and multiplication. [Hint: you will have to use the binary expansion $1/3 = .0101010....$] How many decimal digits of your approximation are correct?

Chapter 2.

1. Suppose that two real numbers r and s are d units apart. Then the larger is $d/2$ units above the midpoint, and the smaller is $d/2$ units below the midpoint. Use these facts to rederive the formulas for the maximum and minimum of two numbers.

2. Using $\max(r, s) = .5(r+s+|r-s|)$, find a formula for $\max(r, s, t)$. Is your formula symmetric in r, s, and t? That is, does reordering the variables change the value of the formula?

3. Apply your formula from the preceding problem to compute $\max(1, 3, 6)$, $\max(1, 6, 3)$, and $\max(6, 1, 3)$, verifying that the same result (6) is obtained in each case.

4. Find a formula for the derivative of $c(x) = \{x$. Is $c(x)$ differentiable at $x = 0$?

5. Suppose the definition of curlyroot is modified to be the inverse of $f(x) = x^3/(\alpha - x)$, so that

$$y = \{x \text{ if and only if } x = \frac{y^3}{\alpha - y},$$

where α is a fixed constant. Show that this modified version of the curlyroot function still leads to a root formula for the equation $x^3 + ax + b = 0$.

6. Let $f(x)$ be a cubic polynomial, say $f(x) = x^3+ax^2+bx+c$. Show that the Newton's iteration function $N(x)$ can be expressed as a rational function $N(x) = p(x)/q(x)$ where p has degree 3 and q has degree 2. Use this fact to transform the equation $N(N(x)) = x$ into a polynomial equation. What is the degree of the equation? What does this tell you about periodic points of period 2 for Newton's method?

7. On page 33 of UME a complicated equation is given for the interpolating polynomial $p(x)$ for the points $(-1, 4), (2, 5), (3, -2)$, and $(4, 1)$. Simplify this expression to find the ascending form for $p(x)$. Then verify that the coefficients satisfy the matrix equation on page 34.

8. Use the method on pages 36 and 37 of UME to find the roots of $p(x) = x^4 + 2x^3 - 5x^2 + 2x + 1$.

9. Use the method on pages 37 and 38 of UME to find the roots of $p(x) = x^6 - 7x^5 + 15x^4 - 14x^3 + 15x^2 - 7x + 1$.

10. Complete the analysis on pages 39 and 40 to find one exact root of equation 6 on page 39.

Define a polynomial $p(x)$ to anti-palindromic if the reverse polynomial equals $-p(x)$. That is, if $p(x)$ has degree n, then p is anti-palindromic if and only the coefficient of x^k is the opposite of the coefficient of x^{n-k}. So for example, $x^5 + 3x^4 + 7x^3 - 7x^2 - 3x - 1$ is anti-palindromic. The next two problems concern anti-palindromic polynomials.

11. Prove that $p(x)$ is anti-palindromic if and only if

$$p(x) = (x - 1)q(x)$$

where $q(x)$ is palindromic.

12. Show that if r is a non-zero root of an anti-palindromic polynomial so is $1/r$.

Chapter 3.

1. Show that u and v satisfy the system of equations

$$\begin{aligned} u + v &= b \\ uv &= c \end{aligned}$$

   if and only if they are the roots of the quadratic equation

$$x^2 - bx + c = 0.$$

2. Show that $\sigma_3(x, y, z, w) = w\sigma_2(x, y, z) + \sigma_3(x, y, z)$. Generalize this result.

3. $f(x, y, z) = x/y + x/z + y/x + y/z + z/x + z/y$ is a symmetric function. Can it be expressed as a combination of the elementary symmetric functions?

4. The difficulties discussed on page 50 of UME of inverting the symmetric function equations are reduced for palindromic polynomials. As an example, find the roots of the palindromic quartic $x^4 + 5x^3 + 6x^2 + 5x + 1$ as follows. Assume that the roots are r, s, and their reciprocals and write out the equations that express the coefficients as symmetric functions of the roots. Then solve these equations to find r and s.

5. Let $f(x, y, z)$ be any function of three variables. Let α, β, and γ be any symmetric functions of the variables $r_1, r_2, \cdots, r_n$. Show that $f(\alpha, \beta, \gamma)$ is a symmetric function of $r_1, r_2, \cdots, r_n$.

6. Does the cubic equation $x^3 - 5x^2 + 3x - 7 = 0$ have distinct roots? [Hint: use equation (5) on page 54 of UME.]

7. Let $p(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$. Use Newton's identities (equations (6) and (7) on page 57 of UME) to find the sum of the roots, the sum of the squares of the roots, the sum of the cubes of the roots, and the sum of the fourth powers of the roots of p.

8. Redo the preceding problem using the long division method (page 61 of UME).

9. Using the same $p(x)$ as in the two preceding problems, find the roots, and then verify that your answers to the two preceding problems are correct. [Hint: to find the roots of p, either use a graphing calculator or guess some likely integers. In either case, confirm that your roots are correct by substitution into p using synthetic division.]

10. Is $\sqrt[5]{7+\sqrt{17}}+\sqrt[5]{7-\sqrt{17}}$ rational?

11. Show that $(1\pm\sqrt{3})^3 = 10\pm 6\sqrt{3}$. Then show that $\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{10+6\sqrt{3}}$ is rational.

12. Is $\sqrt[5]{42+29\sqrt{2}}+\sqrt[5]{42-29\sqrt{2}}$ rational?

Chapter 4.

1. If r is a rational root of $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + 1$ where each a_k is an integer, show that $r = 1/m$ for some integer m which is a divisor of a_n.

2. Show that a polynomial with rational coefficients is a constant multiple of a polynomial with integer coefficients.

3. Find all the roots of
$$p(x) = \frac{1}{6}x^5 - \frac{1}{2}x^4 - \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{2}{3}x + 2.$$

4. Compute $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$ on a calculator. We have seen that the exact value of the expression is 2. Does your calculator give a consistent result?

5. Using a calculator, evaluate the expression $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$. Guess the exact value of the expression, and prove that your guess is correct.

6. Carry out the missing steps on page 74 of UME to derive the expressions $u + v, \omega u + \omega^2 v$, and $\omega^2 u + \omega v$ for the three roots of the cubic. That is, let $u = u_0$ and $v = b/(3u_0)$, and use the fact that the roots of the cubic can be expressed as $u_0 + b/(3u_0), \omega u_0 + b/(3\omega u_0)$, and $\omega^2 u_0 + b/(3\omega^2 u_0)$.

7. Continuing the preceding problem, simplify
$$\begin{aligned} a_0 &= -(u+v)(\omega u + \omega^2 v)(\omega^2 u + \omega v) \\ a_1 &= (u+v)(\omega u + \omega^2 v) + (u+v)(\omega^2 u + \omega v) + (\omega u + \omega^2 v)(\omega^2 u + \omega v) \\ a_2 &= -(u+v) - (\omega u + \omega^2 v) - (\omega^2 u + \omega v). \end{aligned}$$
to derive
$$\begin{aligned} a_0 &= -u^3 - v^3 \\ a_1 &= -3(uv) \\ a_2 &= 0. \end{aligned}$$

8. Does the cubic equation $x^3 = 9x + 11$ have three real roots? [Hint: use the discriminant discussed on page 75 of UME.]

9. Use calculus to analyze the graph of $f(x) = x^3 - bx - c$. Show that if f has three real roots then there must be a positive local maximum and a negative local minimum. [Hint: use Rolle's theorem and the second derivative test.] Then derive the discriminant condition by observing that the product of the max and min values must be negative.

The next four problems concern **Equality of Two Cubes** on page 79 of UME.

10. Show that the equation $x^3 = ax + b$ cannot be written in the form $A(x+m)^3 = B(x+n)^3$ with $m = n$, except in a trivial case.

11. Show that if the equation $x^3 = ax + b$ can be written in the form $A(x+m)^3 = B(x+n)^3$ then A, B, m, and n must satisfy

$$\begin{aligned} A - B &= 1 \\ 3Am - 3Bn &= 0 \\ 3Am^2 - 3Bn^2 &= -a \\ Am^3 - Bn^3 &= -b. \end{aligned}$$

12. Using the first two equations of the system in the preceding problem, express A and B in terms of m and n. Then use those results and the last two equations to derive

$$\begin{aligned} m + n &= \frac{3b}{a} \\ mn &= \frac{a}{3}. \end{aligned}$$

13. Apply the method of Equality of Two Cubes to Cardano's example $x^3 = 20 - 6x$.

14. On page 36 of UME, properties of palindromic polynomials are used to solve $x^4 + 7x^3 - 2x^2 + 7x + 1 = 0$. Attempt to apply any of the methods for quartics in this chapter to solve this equation.