Chapter 5.

1. Let $f(x, y, z) = 2x + 3y - 4z$ and let $g(x, y, z) = x^2 + y^2 + z^2 - 1$. Also, define $F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$.

   a. Find all points (x, y, z) and multipliers λ such that $g(x, y, z) = 0$ and $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

   b. Find all the stationary points of F (i.e., points where all of the partial derivatives of F vanish).

   c. How do the solutions of the two preceding parts of this problem relate to one another?

2. Referring to figure 5.1, create an example where the curves $g = 0$ and $f = M$ are tangent at p, but also cross there. What can you say about optimizing f subject to the constraint $g = 0$ for your example? More generally, what does this example show about the Lagrange Multipliers method?

3. Apply the implicit function theory approach described on page 104 of UME to the problem of minimizing $x^2 + y^2 + z^2$ over the surface $xyz = 1$. In particular, find the functions f, g, and Φ for this example, calculate $d\Phi$, and apply the chain rule to show that $\nabla(f \circ \Phi) = 0 = \nabla(g \circ \Phi)$ implies that ∇f and ∇g are parallel. Hint: it is not necessary to find the solutions to $\nabla(f \circ \Phi) = 0 = \nabla(g \circ \Phi)$. Follow the argument in the book to show that ∇f and ∇g are both orthogonal to the plane spanned by the columns of $d\Phi$.

4. Section 5.3 of UME concerns an example for which F is shown to have neither a local maximum nor a local minimum at the solutions to the Lagrange Multiplier conditions. In that analysis, F is restricted to a plane, reducing it to a function of two variables. For this exercise, show directly that none of the critical points of F is a local maximum or minimum by applying the second derivative test for a function of three variables.

5. Extend the theorem in section 5.4 of UME to functions of n variables.

6. Consider the problem of maximizing $x + y$ for points on the circle $x^2 + y^2 = 2$.

   a. Parameterize the circle as $(x(t), y(t)) = \sqrt{2}(\cos t, \sin t)$ to show that the solution occurs at $(x, y) = (1, 1)$.

   Following the idea of leveling the playing field, we define the family of functions $F_\lambda(x, y) = x + y + \lambda(x^2 + y^2 - 2)$.

   b. Show that at the point $(1, 1)$ the directional derivative of F_λ in the direction of the vector $(-1, 1)$ (tangent to the circle) is 0, independent of the choice of λ.

   c. Show that at the point $(1, 1)$ the directional derivative of F_λ in the direction of the vector $(1, 1)$ (normal to the circle) is $\sqrt{2}(1 + 2\lambda)$, which is zero if and only if $\lambda = -1/2$.

   d. Show that at $F_{-1/2}(x, y)$ has a critical point at $(1, 1)$.

   e. Relate these results to the idea of leveling the playing field.

Chapter 6.

1. A family of lines is defined as follows. For each a, let C_a be the line from $(-a, a)$ to $(6-a, 6-a)$. Find the envelope for this family of lines, and show that it is a parabola with a vertical axis.

2. Consider a string art design with pegs on the x and y axis at points with positive integer coordinates. For a fixed positive integer N, let $p_0 = (N, 0)$, $p_1 = (N-1, 0)$, $p_2 = (N-2, 0)$, and so on. Also, let $q_0 = (0, 0)$, $q_1 = (0, 1)$, $q_2 = (0, 2)$, and so on. If the strings are tied from p_k to q_k where $k = 0, 1, \cdots, N$, show that the envelope is a parabola.

3. As a variation on the preceding problem shift all the p points by a fixed amount δ, so that $p_k = (N-k+\delta, 0)$. Show that the envelope is still a parabola.

4. For the most general case, suppose that the pegs for a string art design are set along two straight lines, intersecting at a point A. Let the end points be equally spaced on each line (although not necessarily with the same spacing on both lines), and arranged so they move successively toward A on one line and away from A on the other. Show that for such a pattern, the envelope is always a parabola.

5. Let S be a given positive constant. Among all polynomials $x^n - Sx^{n-1} + a_{n-2}x^{n-2} + ... + a_0$ with positive real roots, find the one for which $|a_0|$ is maximal.

6. Among all polynomials $x^n - (n+1)x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$ with positive real roots, find the maximal possible value of $|a_0|$. What is the limit of this maximal value as n goes to infinity?

7. Let $\Phi(x, y) = (P, A) = (2x + 2y, xy)$ for $x, y \geq 0$, as in Figure 6.10. Show that in the image of Φ, $0 \leq A \leq P^2/16$, or in terms of x and y, $0 \leq xy \leq (x+y)^2/4$.