Exercises for Uncommon Mathematical Excursions

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July 2009

UME Solutions

Chapter 1.

1. a. To see how this works, you could look at an example. If $p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$ then the reverse polynomial is $5 - 11x + 6x^2 + 7x^3 - 3x^4 = x^4(5/x^4 - 11/x^3 + 6/x^2 + 7/x - 3) = x^4 p(1/x)$.
For a general proof, let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then we have

$$\begin{aligned} x^n p(1/x) &= x^n(a_n/x^n + a_{n-1}/x^{n-1} + \cdots + a_1/x + a_0) \\ &= a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n, \end{aligned}$$

which is the reverse polynomial.

   b. Given a polynomial p, let q be the reverse polynomial. Then we have $p(1/x) = q(x)/x^n$. So, we can mentally evaluate $q(x)$ using Horner form, and then divide by x^n. For the given example, we have $x = 3$ so we are to compute $q(3)$ and divide by $3^4 = 81$. Here, the reverse sequence of coefficients is $-3, 7, 6, -11, 5$. Horner evaluation produces the intermediate results $-9, -2, -6, 0, 0, -11, -33, -28$. Therefore, $p(1/3) = -28/81$.

2. Since the binary digits are coefficients in a polynomial $p(x)$ and we wish to compute $p(2)$, Horner evaluation requires that we repeatedly multiply by 2 and add the next coefficient. Applying this to 1101001 we obtain the following intermediate results: 2, 3, 6, 6, 12, 13, 26, 26, 52, 52, 104, 105. Therefore 1101001 represents 105.

3. In the solution of the preceding problem, the algorithm is to either double each succeeding result, or double and add 1, depending on whether the corresponding binary digit is 0 or 1. To reverse the process, we must either subtract 1 (if the prior result was odd) and divide by 2, or simply divide by 2 (if the prior result was even). Each time we subtract and divide we must record a 1, and when we just divide we record a 0. That will generate the binary digits in reverse order. So, starting with 105, which is odd, we record a 1. Then subtract 1 and divide by 2: 52. That is even, so we record a 0, and divide by 2: 26. Even again, record 0, divide by 2: 13. Odd, record 1, subtract and divide: 6. Even, record 0, divide: 3. Odd, record 1, subtract and divide 1. Record that

final 1 and stop. So, we have recorded 1, 0, 0, 1, 0, 1, 1. Reverse the digits to obtain the binary form of 105, 1101001.

4. First, express 36 in binary, using the method of the preceding problem: 100100. Next, apply the process from page 12 of UME to get

$$1.005^{36} = 1.005^{((((2)2)2+1)2)2} = ((((1.005^2)^2)^2 \times 1.005)^2)^2.$$

If you put 1.005 in memory, then a valid sequence of operations to compute 1.005^{36} is: recall, square, square, square, times, recall, equals, square square equals.

5. First, use synthetic division or Horner evaluation to divide $p(x) - p(a)$ by $x - a$. Actually, we don't have to do the complete division since we know the remainder will be 0. So just complete the division algorithm for the first 3 steps, generating the coefficients 5, $5a + 6$, $5a^2 + 6a - 2$, and hence the quotient $5x^2 + (5a + 6)x + (5a^2 + 6a - 2)$. Now change a to x and simplify: $5x^2 + (5x + 6)x + (5x^2 + 6x - 2) = 15x^2 + 12x - 2$. That is $p'(x)$.

6. Approximating 1/3 by the binary expansion 0.01 is equivalent to approximating the cuberoot by a fourth root. To compute this we simply enter 1.23 and hit the squareroot key twice, or in symbols, 1.23, $\sqrt{\ }$, $\sqrt{\ }$, =. Using 0.0101 we get this key sequence: 1.23, $\sqrt{\ }$, $\sqrt{\ }$, =, ×1.23, =, $\sqrt{\ }$, $\sqrt{\ }$, =. For each additional 01 in the binary expansion, the computation requires an additional block ×1.23, = $\sqrt{\ }$, $\sqrt{\ }$, =, in the key sequence. Using 0.01010101, for example, produces an estimate for $1.23^{1/3}$ that is accurate to three decimals. For 5 decimal place accuracy, seven cycles are required.

Chapter 2.

1. The distance between r and s is $d = |r - s|$. The midpoint is $m = (r + s)/2$. Thus, the larger of the two numbers is

$$m + \frac{d}{2} = \frac{r + s + |r - s|}{2},$$

and the smaller is

$$m - \frac{d}{2} = \frac{r + s - |r - s|}{2}.$$

2. Observe that $\max(r, s, t) = \max(r, \max(s, t))$. This leads to

$$\max(r, s, t) = \frac{2r + s + t + |s - t| + |s + t + |s - t| - 2r|}{4}.$$

This is not obviously symmetric. That is, it is difficult to see by looking at the form of the expression that reordering the variables leaves the value of the function unchanged. Yet we know that is true because $\max(r, s, t)$ is a symmetric function.

3. For the first computation, we have

$$\begin{aligned} \max(1, 3, 6) &= \frac{2 + 3 + 6 + |3 - 6| + |3 + 6 + |3 - 6| - 2|}{4} \\ &= \frac{11 + 3 + |9 + 3 - 2|}{4} \\ &= \frac{14 + 10}{4} \\ &= 6 \end{aligned}$$

The other computations are similar.

4. Differentiating identity (3) on page 27 of UME gives

$$3\,\{\overline{x}^{\,2}\,(\{\overline{x})' = 1 - \{\overline{x} - x\,(\{\overline{x})'.$$

Rearrangement then leads to

$$(\{\overline{x})' = \frac{1 - \{\overline{x}}{x + 3\,\{\overline{x}^{\,2}}$$

This formula is not valid at $x = 0$. In fact, we can verify that the curlyroot is not differentiable at 0 by considering the inverse function, $x^3/(1-x)$. Since it has slope 0 at $(0,0)$, the tangent line to the curve is horizontal there. Therefore the graph of $c(x)$ has a vertical tangent line at $(0,0)$, and so is not differentiable at $x = 0$.

5. With the modified definition, a root of $x^3 + ax + b = 0$ is given by

$$r = -\frac{b}{\alpha a} \left\{ \overline{\frac{\alpha^2 a^3}{b^2}} \right. .$$

The verification of this formula is similar to the one given on the bottom of page 27 of UME.

6. $N(x) = x - f(x)/f'(x) = (2x^3 + ax^2 - c)/(3x^2 + 2ax + b)$. Taking $p(x) = 2x^3 + ax^2 - c$ and $q(x) = 3x^2 + 2ax + b$ completes the first part of the problem. Now observe that $N(N(x)) = N(p/q)$ can be simplified as follows:

$$\begin{aligned} N(N(x)) &= \frac{2p^3/q^3 + ap^2/q^2 - c}{3p^2/q^2 + 2ap/q + b} \\ &= \frac{2p^3 + ap^2q - cq^3}{3p^2q + 2apq^2 + bq^3} \\ &= \frac{p_2}{q_2} \end{aligned}$$

where $p_2(x)$ is a polynomial of degree 9 (with leading term $16x^9$) and $q_2(x)$ is a polynomial of degree 8 (with leading term $36x^8$). Therefore the equation $N(N(x)) = x$ can be written as

$$p_2(x) - xq_2(x) = 0.$$

That is a polynomial of degree 9. Its roots are the fixed points of $N(N(x))$ and so are the period 2 points for the iteration of $N(x)$. If all the roots are real we will obtain 9 points of period 2 (including up to 3 roots of f which are fixed points of N).

7. Simplification shows that

$$p(x) = \frac{81}{5} + \frac{53}{15}x - \frac{73}{10}x^2 + \frac{41}{30}x^3.$$

Thus, the coefficients are $a = 81/5$, $b = 53/15$, $c = -73/10$, and $d = 41/30$. Substitution in the matrix equation leads to

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} 81/5 \\ 53/15 \\ -73/10 \\ 41/30 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

which is indeed correct.

8. With u, v, r, and s defined as on page 36, we are led to

   the equations $u + v = -2$ and $uv = -7$. They tell us that u and v are the roots of the quadratic $x^2 + 2x - 7$, and so are given by $-1 \pm 2\sqrt{2}$. This in turn gives us the factorization

   $$p(x) = [x^2 + (1 - 2\sqrt{2})x + 1][x^2 + (1 + 2\sqrt{2})x + 1],$$

   so we can find the roots of p by finding the roots of the quadratic factors. For the first factor the roots are

   $$\frac{-1 + 2\sqrt{2} \pm i\sqrt{4\sqrt{2} - 5}}{2}$$

   while the roots of the second factor are

   $$\frac{-1 - 2\sqrt{2} \pm \sqrt{5 + 4\sqrt{2}}}{2}.$$

9. Here, the coefficients are $a = -7$, $b = 15$, and $c = -14$. After introducing $u = x + 1/x$ we obtain the cubic equation

   $$u^3 - 7u^2 + 12u = 0.$$

   This has roots of 0, 3, and 4. To find the roots of the original equation, we must now solve three equations:

   $$\begin{aligned} x + 1/x &= 0 \\ x + 1/x &= 3 \\ x + 1/x &= 4, \end{aligned}$$

   or, after rearrangement

   $$\begin{aligned} x^2 + 1 &= 0 \\ x^2 - 3x + 1 &= 0 \\ x^2 - 4x + 1 &= 0. \end{aligned}$$

   Therefore, the roots of the original equation are $\pm i$, $(3 \pm \sqrt{5})/2$, and $2 \pm \sqrt{3}$.

10. One root is given by

$$\frac{-3-\sqrt{57}+\sqrt{50+6\sqrt{57}}+i\sqrt{(6+2\sqrt{57})\sqrt{50+6\sqrt{57}}-52-12\sqrt{57}}}{8}.$$

11. Suppose that $p(x)$ is anti-palindromic and of degree n. Then $p(1)$, which is the sum of the coefficients, must be 0, because the coefficient of x^k is cancelled by the coefficient of x^{n-k}. If $n = 2m$ is even, note that x^m and x^{n-m} are one and the same, so in this case the coefficient of x^m is 0.

    Since $p(1) = 0$ we can factor $p(x) = (x-1)q(x)$ for a polynomial q. We must show that $q(x) = p(x)/(x-1)$ is palindromic. To that end, consider the reverse polynomial

$$\begin{aligned} x^{n-1}q(1/x) &= \frac{x^{n-1}p(1/x)}{1/x-1} \\ &= \frac{x^n p(1/x)}{1-x} \\ &= -\frac{x^n p(1/x)}{x-1}. \end{aligned}$$

    But $x^n p(1/x)$ is the reverse polynomial for p, and that must be $-p(x)$ because p is anti-palindromic. Therefore, we see

$$\begin{aligned} x^{n-1}q(1/x) &= \frac{p(x)}{x-1} \\ &= q(x). \end{aligned}$$

    This shows that the reverse polynomial of q equals q, and so q is palindromic.

    For the converse, suppose $p(x) = (x-1)q(x)$ where we know q is palindromic. Then the reverse polynomial for p is

$$\begin{aligned} x^n p(1/x) &= x^n(1/x-1)q(1/x) \\ &= x^{n-1}(1-x)q(1/x) \\ &= -(x-1)x^{n-1}q(1/x) \\ &= -(x-1)q(x) \end{aligned}$$

    where the last equality follows because q is palindromic of degree $n-1$. Thus we have shown that the reverse polynomial of p is $-p$, and that implies that p is anti-palindromic. This completes the proof.

12. If $r = 1$ then $1/r = r$ so in this case $1/r$ is a root if r is. So consider a root r different from 1. By virtue of the factorization $p(x) = (x - 1)q(x)$ established in the prior problem, we see that r must be a root of the palindromic polynomial q. But that implies $1/r$ is also a root of q, and hence of p.

Chapter 3.

1. If u and v are the roots of $x^2 - bx + c = 0$ then we must have the identity $(x-u)(x-v) = x^2 - bx + c$. Equating coefficients on each side of the identity shows that $u + v = b$ and $uv = c$.

   Conversly, assume u and v satisfy the system. Then, multiplying the first equation by u leads to

   $$u^2 + uv = bu,$$

   and substituting $uv = c$ we obtain

   $$u^2 - bu + c = 0.$$

   Thus u is a root of the specified quadratic. A similar argument shows that v is also a root of the quadratic. Now if u and v are distinct, we infer that they are precisely the roots of the quadratic. On the other hand, if $v = u$, the system reduces to the equations $2u = b$ and $u^2 = c$. Combining these leads to $b^2 = 4c$, indicating that $x^2 - bx + c$ is a perfect square and so has a single root. In this case, as well, u and v are precisely the roots of the quadratic.

2. Beginning with $\sigma_3(x, y, z, w) = yzw + xzw + xyw + xyz$, factor w out of the first three terms. That gives $\sigma_3(x, y, z, w) = w(yz + xz + xy) + xyz = w\sigma_2(x, y, z) + \sigma_3(x, y, z)$. More generally, we can group the terms of $\sigma_j(x_1, \cdots, x_n)$ into two sets, those that contain x_n and those that do not. For the sum of the terms in the first set, we can factor out the common x_n and what remains must be $\sigma_{j-1}(x_1, \cdots, x_{n-1})$. The remaining terms are all possible products of k variables chosen from $\{x_1, \cdots, x_{n-1}\}$, and so sum to $\sigma_k(x_1, \cdots, x_{n-1})$. This shows that $\sigma_k(x_1, \cdots, x_n) = x_n\sigma_{k-1}(x_1, \cdots, x_{n-1}) + \sigma_k(x_1, \cdots, x_{n-1})$

3. Yes: $\sigma_1\sigma_2/\sigma_3 - 3$

4. We assume that the roots of $p(x) = x^4 + 5x^3 + 6x^2 + 5x + 1$ are $r, 1/r, s$, and $1/s$. Expressing the coefficients in terms of symmetric functions of these roots therefore gives

   $$\begin{aligned} r + 1/r + s + 1/s &= -5 \\ 1 + rs + r/s + s/r + 1/rs + 1 &= 6 \\ s + 1/s + r + 1/r &= -5 \\ 1 &= 1 \end{aligned}$$

Clearly, the last two equations are redundant, so we focus on solving the first two for r and s. Rewrite the second equation as

$$rs + r/s + s/r + 1/rs = 4$$

and notice that the left side factors, giving

$$(r + 1/r)(s + 1/s) = 4.$$

Combined with the original first equation, this suggests introducing

$$\begin{aligned} u &= r + 1/r \\ v &= s + 1/s \end{aligned}$$

to transform our system into

$$\begin{aligned} u + v &= -5 \\ uv &= 4. \end{aligned}$$

This system we recognize at once as an instance of the sum and product of two unknowns. Accordingly, its solutions are the roots of the quadratic $x^2 + 5x + 4$, namely, -4 and -1. Reintroducing the original variables, we set

$$\begin{aligned} r + 1/r &= -4 \\ s + 1/s &= -1. \end{aligned}$$

These in turn lead to the quadratic equations

$$\begin{aligned} r^2 + 4r + 1 &= 0 \\ s^2 + s + 1 &= 0. \end{aligned}$$

Solving those for r and s gives us the solutions to the original quartic as $-2 \pm \sqrt{3}$ and $(-1 \pm i\sqrt{3})/2$.

5. Let $g(r_1, r_2, \cdots, r_n) = f(\alpha, \beta, \gamma)$. Consider a permutation π of the r's. Since α, β, and γ are symmetric functions, we have

$$\begin{aligned} \alpha(\pi(r_1, \cdots, r_n)) &= \alpha(r_1, \cdots, r_n) \\ \beta(\pi(r_1, \cdots, r_n)) &= \beta(r_1, \cdots, r_n) \\ \gamma(\pi(r_1, \cdots, r_n)) &= \gamma(r_1, \cdots, r_n). \end{aligned}$$

Therefore

$$\begin{aligned} g(\pi(r_1,\cdots,r_n)) &= f(\alpha(\pi(r_1,\cdots,r_n)),\beta(\pi(r_1,\cdots,r_n)),\gamma(\pi(r_1,\cdots,r_n))) \\ &= f(\alpha(r_1,\cdots,r_n),\beta(r_1,\cdots,r_n),\gamma(r_1,\cdots,r_n)) \\ &= g(r_1,\cdots,r_n). \end{aligned}$$

This shows that g is a symmetric function of the r's.

6. We compute the discriminant of $x^3 - 5x^2 + 3x - 7$ using the coefficients $a_2 = -5$, $a_1 = 3$, and $a_0 = -7$. The result is

$$D = 225 - 3500 + 1890 - 108 - 1323 = -2816.$$

Since this is not zero, the cubic does not have any repeated roots.

7. We know that the sum of the roots is the negative of the coefficient of the cubic term, so that is $s_1 = 5$. We also know that s_0 is the sum of the zeroth powers of the roots, and therefore $s_0 = 4$. Also, we know $n = 4$ and the coefficients are given by $a_0 = -6$, $a_1 = 5$, $a_2 = 5$, $a_3 = -5$, and $a_4 = 1$. Using equation (7) with $k = 2$ we get

$$a_4 s_2 + a_3 s_1 + a_2 s_0 = 2a_2.$$

Substituting the known values identified above, this becomes

$$s_2 - 5 \cdot 5 + 5 \cdot 4 = 2 \cdot 5$$

so $s_2 = 15$. Similarly, with $k = 3$ we have

$$a_4 s_3 + a_3 s_2 + a_2 s_1 + a_1 s_0 = a_1$$

and that leads to $s_3 = 35$. Finally, to find s_4, we use equation 6 with $k = 4$. That gives us

$$a_4 s_4 + a_3 s_3 + a_2 s_2 + a_1 s_1 + a_0 s_0 = 0$$

from which we find $s_4 = 99$.

8. To use the long division method, we have to divide the reverse of p into the reverse of the derivative p'. So set up a long division problem to divide $1 - 5x + 5x^2 + 5x^3 - 6x^4$ into $4 - 15x + 10x^2 + 5x^3$. After the first 5 iterations of the long division algorithm the quotient begins $4 + 5x + 15x^2 + 35x^3 + 99x^4$. This shows that $s_0 = 4$, $s_1 = 5$, $s_2 = 15$, $s_3 = 35$, and $s_4 = 99$, verifying the results of the preceding problem.

9. The roots of $p(x)$ are 1, -1, 2, and 3. By direct calculation we find the sum of the roots is 5, the sum of the squares is 15, the sum of the cubes is 35, and the sum of the fourth powers is 99.

10. Let α and β equal $\sqrt[5]{7+\sqrt{17}}$ and $\sqrt[5]{7-\sqrt{17}}$, respectively. We wish to determine whether $r = \alpha + \beta$ is rational. Observe that $\alpha\beta = \sqrt[5]{49-17} = 2$. Therefore, α and β are the roots of the quadratic $x^2 - rx + 2$. Using Newton's identities we can therefore find $s_7 = \alpha^7 + \beta^7$ in terms of r. But we also can compute $\alpha^7 + \beta^7 = 14$. In this we obtain a polynomial that r must satisfy. Alternatively, we can obtain this polynomial by direct algebraic manipulation as follows:

$$\begin{aligned} r^5 &= (\alpha+\beta)^5 \\ &= \alpha^5 + \beta^5 + 5\alpha\beta(\alpha^3+\beta^3) + 10\alpha^2\beta^2(\alpha+\beta) \\ &= 14 + 10(\alpha^3+\beta^3) + 40(\alpha+\beta) \\ &= 14 + 10(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2 + 4) \\ &= 14 + 10r(\alpha^2 + 2\alpha\beta + \beta^2 + 4 - 3\alpha\beta) \\ &= 14 + 10r(r^2 - 2) \\ &= 10r^3 - 20r + 14. \end{aligned}$$

By either approach, we find that r is a root of $r^5 - 10r^3 + 20r - 14$. If it is rational, then it must be a rational root, and therefore an integer divisor of 14. By using simple estimation, we can see that r must be between 1 and 4. So, if r is rational, it must equal 2. Since 2 is not a root of $r^5 - 10r^3 + 20r - 14$, $r \neq 2$, and this shows that r cannot be rational.

11. From the first part of the problem we know that $\sqrt[3]{10\ pm 6\sqrt{3}} = 1 \pm \sqrt{3}$. Thus $\sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10+6\sqrt{3}} = 2$ which is rational.

12. Yes: $\sqrt[5]{42 \pm 29\sqrt{2}} = 1 \pm \sqrt{2}$. Therefore $\sqrt[5]{42+29\sqrt{2}} + \sqrt[5]{42-29\sqrt{2}} = 2$.

Chapter 4.

1. By the rational roots theorem, if r is a rational root, then we can write $r = p/q$ where p and q are integers and p is a divisor of 1, the constant coefficient of the polynomial, and q is a divisor of a_n. This implies that $p = \pm 1$, so $r = 1/m$ where $m = q$ if $p = 1$ and $m = -q$ if $p = -1$. In either case, m is a divisor of a_n, as required.

2. Let the original polynomial be $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Each coefficient can be expressed as a fraction, so write $a_k = p_k/q_k$ for each k, where p_k and q_k are integers. Let m be the least common multiple of $q_0, q_1, \cdots, q_n$. Then $ma_k = mp_k/q_k$ is an integer. This shows that $mf(x)$ is a polynomial with integer coefficients. Since $f(x) = (1/m) \cdot mf(x)$, we have shown that $f(x)$ is a constant multiple of a polynomial with integer coefficients.

3. Multiply both sides of the equation
$$\frac{1}{6}x^5 - \frac{1}{2}x^4 - \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{2}{3}x + 2 = 0$$
by 6 to obtain
$$x^5 - 3x^4 - 3x^3 + 9x^2 - 4x + 12 = 0.$$
Here the polynomial has integer coefficients, so we consider the possibility of rational roots. Since this is a monic polynomials, the only possible rational roots are integer divisors of 12, and so elements of the set $\{-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12\}$. An efficient way to check which if any of these are roots is by synthetic division. For example, we know 2 is a root if and only if $x - 2$ is a factor of the polynomial. Dividing by $x - 2$ we find the factorization
$$x^5 - 3x^4 - 3x^3 + 9x^2 - 4x + 12 = (x-2)(x^4 - 4x^3 - 5x^2 - 4x - 6).$$
We can now proceed in a similar way to seek roots of the quartic factor above, repeatedly considering lower degree factors as each root is found. This leads us to
$$x^5 - 3x^4 - 3x^3 + 9x^2 - 4x + 12 = (x-2)(x+2)(x-3)(x^2+1).$$
This shows that the roots of the original equation are -2, 2, 3, i and $-i$.

4. On a TI-83 plus using an exponent of 1/3 to obtain a cube root, I get an answer of 2. And when I compute $\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{10-6\sqrt{3}}-2$ I get 0 as well. This shows that the calculator's internal representation of $\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{10-6\sqrt{3}}$ is identical to its internal representation of 2.

5. The calculator shows a result of 1. We can prove this is correct as follows. Let $r=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$. Then, using the identity $(u+v)^3=u^3+v^3+3uv(u+v)$, we obtain

$$\begin{aligned} r^3 &= 2+\sqrt{5}+2-\sqrt{5}+3r\sqrt[3]{4-5} \\ &= 4-3r. \end{aligned}$$

This shows that r is a root of the polynomial $f(x)=x^3+3x-4$. But $f'(x)=3x^2+3>0$ for all x, so f is increasing everywhere. This implies that f has a unique real root. On the one hand we know that r is that root. On the other we can see by direct computation that 1 is a root. Therefore, by uniqueness, $r=1$.

6. By definition, the first root, $u_0+b/(3u_0)=u+v$. Next,

$$\begin{aligned} \omega u_0+b/(3\omega u_0) &= \omega u_0+b\omega^3/(3\omega u_0) \\ &= \omega u_0+\omega^2 b/(3u_0) \\ &= \omega u+\omega^2 v. \end{aligned}$$

Similar steps verify the third root expression.

7. Begin with

$$\begin{aligned} a_0 &= -(u+v)(\omega u+\omega^2 v)(\omega^2 u+\omega v) \\ &= -(\omega^3u^3+(\omega^2+\omega^3+\omega^4)u^2v+(\omega^2+\omega^3+\omega^4)uv^2+\omega^3v^3) \\ &= -(u^3+\omega^2(1+\omega+\omega^2)u^2v+\omega^2(1+\omega+\omega^2)uv^2+v^3) \\ &= -u^3-v^3. \end{aligned}$$

The other verifications are similar.

8. For this problem we have $b=9$ and $c=11$. Thus, the discriminant is $c^2-4b^3/27=121-108>0$. This implies that there is a single real root.

9. If f has three real roots, then between the first and second there must be a point where $f'(x)=0$, and likewise between the second and third, by Rolle's theorem. Now $f'(x)=3x^2-b$ has

roots at $x = \pm\sqrt{b/3}$, so b is positive and f has two critical points separated by $x = 0$. Continuing, $f''(x) = 6x$ has the same sign as x, so that f has a maximum and minimum, respectively, at $-\sqrt{b/3}$ and $\sqrt{b/3}$ by the second derivative test. Now if $f(x)$ is positive at both of these points, there can only be one x intercept on the graph, contradicting the assumption of three real roots. Likewise if $f(x)$ is negative at both points. Therefore, the maximum value must be positive and the minimum must be negative. Consequently, the product $f(\sqrt{b/3})f(-\sqrt{b/3}) < 0$. Substituting we obtain $f(\pm\sqrt{b/3}) = -(c \pm (2b/3)\sqrt{b/3})$. Therefore

$$\begin{aligned} f(\sqrt{b/3})f(-\sqrt{b/3}) &= (c + (2b/3)\sqrt{b/3})(c - (2b/3)\sqrt{b/3}) \\ &= c^2 - (4b^2/9)(b/3) \\ &= c^2 - 4b^3/27. \end{aligned}$$

Thus we have shown that if there are three real roots the discriminant must be less than 0. Conversely, if the discriminant is less than 0, b must be positive. Then we can follow the algebra in the opposite direction to conclude that f has a positive local max and a negative local min, and this in turn implies that f has three intercepts.

10. If $m = n$, the equation simplifies to $(A-B)(x+m)^3 = 0$. This has a nontrivial x^2 term, and so cannot be the same as $x^3 = ax + b$, except in the trivial case $m = n = 0 = a = b$.

11. Expanding the binomial cube in each expression gives us

$$A(x+m)^3 = Ax^3 + 3Amx^2 + 3Am^2x + Am^3$$

and

$$B(x+n)^3 = Bx^3 + 3Bnx^2 + 3Bn^2x + Bn^3.$$

Substituting in the equation $A(x+m)^3 - B(x+n)^3 = 0$ and equating coefficients with $x^3 - ax - b = 0$ gives the desired result.

12. Divide the second equation by 3 and multiply the first equation by m, obtaining

$$\begin{aligned} Am - Bm &= m \\ Am - Bn &= 0. \end{aligned}$$

Subtracting now leads to $B = m/(n-m)$. A similar procedure can be used to eliminate B from the two original equations, resulting in $A = n/(n-m)$.

Now substitute those expressions in

$$\begin{aligned} Am^2 - Bn^2 &= -a/3 \\ Am^3 - Bn^3 &= -b \end{aligned}$$

to obtain

$$\begin{aligned} \frac{m^2n - mn^2}{n - m} &= -a/3 \\ \frac{m^3n - mn^3}{n - m} &= -b. \end{aligned}$$

Factoring leads to

$$\begin{aligned} \frac{mn(m - n)}{n - m} &= -a/3 \\ \frac{mn(m - n)(m + n)}{n - m} &= -b. \end{aligned}$$

Simplification now produces

$$\begin{aligned} mn &= a/3 \\ mn(m + n) &= b. \end{aligned}$$

As a final step, replace mn in the second equation with $a/3$ to reach the desired conclusion:

$$\begin{aligned} mn &= a/3 \\ m + n &= 3b/a. \end{aligned}$$

13. First, note that in this problem $a = -6$ and $b = 20$. The equations for m and n therefore become

$$\begin{aligned} mn &= -2 \\ m + n &= -10, \end{aligned}$$

so we conclude that m and n are the roots of $t^2 + 10t - 2 = 0$. Using the quadratic formula, we define $m = -5 - 3\sqrt{3}$ and $n = -5 + 3\sqrt{3}$. These in turn give us

$$\begin{aligned} A &= \frac{-5 + 3\sqrt{3}}{6\sqrt{3}} \\ B &= \frac{-5 - 3\sqrt{3}}{6\sqrt{3}}. \end{aligned}$$

We can now find a root using the identity

$$x = \frac{B^{1/3}n - A^{1/3}m}{A^{1/3} - B^{1/3}},$$

obtaining

$$x = \frac{(5-3\sqrt{3})\sqrt[3]{5+3\sqrt{3}} - (5+3\sqrt{3})\sqrt[3]{5-3\sqrt{3}}}{\sqrt[3]{5+3\sqrt{3}} - \sqrt[3]{5-3\sqrt{3}}}.$$

This can be algebraically manipulated into Cardano's solution $\sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}}$.

14. My attempts to solve the quartic using the methods of Chapter 4 all bogged down in a hopeless algebraic morass. I would be grateful to hear from anyone who has successfully applied one or more of those methods.