The latex source file for this note was produced by the Google AI agent, by means of a series of prompts from Dan Kalman. Added annotations appear in red boxes. Some are meant to clarify contributions from the AI, including some that have not been verified by Kalman. To repeat the warning that appears each time the AI responds to a prompt: "AI can make mistakes, so double-check responses" .

Binomial Expansions and Commutativity of Matrices

Abstract

Let $A, B \in M_m(\mathbb{C})$ and $S = A + B$. We show that if S is invertible and the binomial expansion holds for S^n, S^{n+1}, and S^{n+2} for some integer $n > 2$, then A and B must commute. The proof is elementary, relying only on binomial identities, Pascal's identity, and basic matrix algebra. We also comment on a stronger result, where the binomial expansion for just S^n and S^{n+1} suffices; that version depends on deeper commutative algebra and functional identities. Finally, we note that the main theorem of this paper, while weaker than the commutative-algebra result, is still of independent interest and is provable by elementary means.

Let A and B be $m \times m$ matrices. As is well known, when $AB = BA$ the binomial theorem holds. That is, for any positive integer n,

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k. \tag{1}$$

In general this equation does not hold when $AB \neq BA$. It is natural to ask whether the binomial expansion is valid *only* when A and B commute. One immediately sees that (1) with $n = 2$ implies $AB = BA$. On the other hand, one can construct examples for which the equation holds with $n = 3$ with $AB \neq BA$. In one such example with 3×3 matrices, A is diagonal with diagonal entries 1, -2, and 4, and B has 1's on the superdiagonal and 0's elsewhere.

The purpose of this paper is to show by elementary means that commutativity of A and B obtains if (1) holds for three consecutive values of n. The formal statement of this result follows.

The theorem below is valid for n > 0. If n = 1 or 2, then 2 is included in the set {n, n+1, n+2} and assuming the binomial expansion of (A+B)^2 implies AB = BA.

Theorem 1. *Let $A, B \in M_m(\mathbb{C})$ and set $S = A+B$. Assume S is invertible. Suppose that for some integer $n > 2$ the binomial expansions*

$$S^k = (A+B)^k = \sum_{j=0}^{k} \binom{k}{j} A^{k-j} B^j$$

hold for each $k \in \{n, n+1, n+2\}$. Then $AB = BA$.

Proof. We divide the proof into three steps.

Step 1: Using S^n and S^{n+1} to show A and B commute with S^n. Since the binomial expansion holds for S^n, we have

$$S^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k. \tag{2}$$

Multiplying (2) on the left by A gives

$$AS^n = \sum_{k=0}^{n} \binom{n}{k} A^{n+1-k} B^k. \tag{3}$$

Multiplying (2) on the right by B gives

$$S^n B = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^{k+1}.$$

Reindex this sum by setting $j = k+1$, so $k = j-1$ and j runs from 1 to $n+1$:

$$S^n B = \sum_{j=1}^{n+1} \binom{n}{j-1} A^{n+1-j} B^j. \tag{4}$$

Adding (3) and (4) yields

$$AS^n + S^n B = \sum_{k=0}^{n} \binom{n}{k} A^{n+1-k} B^k + \sum_{k=1}^{n+1} \binom{n}{k-1} A^{n+1-k} B^k.$$

We now combine these into a single sum over $k = 0, 1, \ldots, n+1$. For $1 \le k \le n$, the coefficient of $A^{n+1-k} B^k$ is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Step 1 is due entirely to the AI. It included this argument in support of an assertion it made, independent of ideas already discussed in our dialog. The argument took Kalman completely by surprise. Although the supported assertion turned out to have a logical error, Step 1 is itself valid. What appears here is essentially identical to what appeared on the computer screen during the dialog with the AI, complete with summation notation and manipulations, binomial coefficients, and the Pascal's matrix identity.

by Pascal's identity. For $k = 0$, only the first sum contributes, giving $\binom{n}{0} = \binom{n+1}{0}$. For $k = n+1$, only the second sum contributes, giving $\binom{n}{n} = \binom{n+1}{n+1}$.

Thus

$$AS^n + S^n B = \sum_{k=0}^{n+1} \binom{n+1}{k} A^{n+1-k} B^k = S^{n+1}, \tag{5}$$

where the last equality uses the assumed binomial expansion for S^{n+1}.

On the other hand,

$$S^{n+1} = (A+B)S^n = AS^n + BS^n.$$

Comparing with (5) gives

$$BS^n = S^n B.$$

Similarly,

$$S^{n+1} = S^n(A+B) = S^n A + S^n B,$$

and comparing with (5) gives

$$S^n A = AS^n.$$

Thus we have shown

$$AS^n = S^n A, \qquad BS^n = S^n B.$$

Step 2: Using S^{n+1} and S^{n+2} to show A and B commute with S^{n+1}.

Repeating the same argument with n replaced by $n+1$ (which is valid since $n+1 > 3$), the binomial expansions for S^{n+1} and S^{n+2} imply

$$S^{n+2} = AS^{n+1} + S^{n+1}B.$$

Comparing with

$$S^{n+2} = (A+B)S^{n+1} = AS^{n+1} + BS^{n+1}$$

and with

$$S^{n+2} = S^{n+1}(A+B) = S^{n+1}A + S^{n+1}B,$$

we obtain

$$AS^{n+1} = S^{n+1}A, \qquad BS^{n+1} = S^{n+1}B.$$

Step 3: Commuting with S^n and S^{n+1} implies commuting with S.

We use the following lemma.

Originally, the AI assumed the binomial expansion for *two* consecutive integers, n and n+1, and derived the fact that S^n commutes with A and B. It then fallaciously used the argument in the lemma below, as if S^{n+1} had also been shown to commute with A and B. Kalman's contribution was to notice this error, and extend the assumed validity of the binomial expansion from 2 to 3 consecutive integers. The lemma and its use to complete the proof of the theorem are due to the AI.

Lemma. Let S be invertible and X a matrix such that

$$XS^n = S^n X \quad \text{and} \quad XS^{n+1} = S^{n+1}X.$$

Then $XS = SX$.

Proof of lemma. Since S and S^n commute, $S^{n+1} = S^n S = SS^n$. Using $XS^n = S^n X$, we compute

$$XS^{n+1} = XS^n S = S^n XS.$$

On the other hand,

$$S^{n+1}X = S^n SX.$$

Thus

$$S^n XS = S^n SX.$$

Multiplying on the left by $(S^n)^{-1}$ gives

$$XS = SX.$$

$\square$

Applying the lemma to $X = A$ and $X = B$, and using the commutation relations established in Steps 1 and 2, we conclude

$$AS = SA, \qquad BS = SB.$$

Finally, since $S = A + B$, the identity $AS = SA$ becomes

$$A(A + B) = (A + B)A,$$

which expands to

$$A^2 + AB = A^2 + BA.$$

Cancelling A^2 yields

$$AB = BA.$$

This completes the proof. $\square$

The stronger result stated below and the references have not been verified as of this writing. Kalman does not know whether the references actually exist, nor whether they include results that imply the two-consecutive-integer version of the main theorem here, along the lines described below.

Additional Remarks

Stronger result using only two exponents

Theorem 1 assumes that the binomial expansion holds for three consecutive exponents $n, n+1, n+2$. A stronger result is also true: if $n > 2$ and the binomial expansion holds for just S^n and S^{n+1}, then A and B must already commute.

The proof of this stronger statement is not elementary. From the two binomial identities one derives a functional identity of the form

$$\sum_{j=0}^{n-2} S^j (AB - BA) S^{n-2-j} = 0,$$

and results of Brešar and collaborators on functional identities in prime rings imply that the only solution is $AB - BA = 0$. See:

- M. Brešar, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings*, Trans. Amer. Math. Soc. **335** (1993), 525–546.
- M. Brešar, M. A. Chebotar, and W. S. Martindale 3rd, *Functional Identities*, Birkhäuser, 2007.

Elementary nature of the main theorem

Although Theorem 1 is weaker than the functional-identity result, it has the advantage of being provable by entirely elementary methods. No ring-theoretic machinery is required; the argument uses only:

- the binomial theorem for matrices under the stated hypotheses,
- Pascal's identity,
- basic manipulations with invertible matrices.

Thus the main theorem stands as an accessible result that illustrates how binomial identities constrain the algebraic structure of matrix pairs.